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Diploma thesis

Collision-dominated spin transport in graphene

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Acknowledgment

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Trieste, August 26, 2010.

Contents

1 Introduction

Graphene has attracted a lot of interests in recent years. However the discussion of the band structure of such a system was actually initiated long ago by P. R. Wallace when he studied electrical properties of graphite.[1] This was far before the birth of low-dimensional physics, and its attractive properties did not receive much attention at the same time. Recently, with advances in low-dimensional physics, and especially in synthesizing Carbon based materials, in 2004, K. S. Novoselov *et al.* isolated graphene mono-layers for the first time. It soon became one of the central interests between condensed-matter theoreticians as well as experimentalists.[2] Graphene exhibits a lot of intriguing aspects from the theoretical point of view. At the same time, the physical properties of the material make it an important candidate for applications in modern technology. Despite a lot of works, many aspects of this novel material are still unsolved. Recently, a nice review of the general topic was given by an article of A. H. Castro Neto *et al.*.[3]

Graphene is simply a honey-comb lattice of Carbon atoms in a plane. The Bravais lattice of such a crystal is triangular with two atoms per unit cell. Three of the four valence electrons in the outer shell of the Carbon atoms go into sp^2 hybridized orbitals, which are responsible for the stability of the honey-comb crystal. The fourth one is still in the p_z orbital. They actually form a so-called *super combination* of π electron over all the lattice, and are responsible for most of the electronic and optical properties of graphene. Qualitatively, the band structure of graphene can be studied by the simple tight-binding approximation (two orbitals per unit cell), which is shown in Fig. 1. The lower band is filled by the π electrons, while the upper band is empty. The two bands touch at the six corners of the hexagonal Brillouin zone. However only two of these corners are independent, giving rise to the two Dirac points K and K', which is actually the Fermi surface (or rather Fermi points) of the system. Moreover, the band structure near the Fermi surface is linear, being reminiscent of the relativistic dispersion relation of light or other massless particles. Indeed, low energy excitations of electron in such a system were longed to obey the two-dimensional Dirac equation, where the Fermi velocity v_F plays the role of the speed of light.[3] This results in many anomalous behaviors of graphene. For example, one of the interesting effects is Klein-tunneling, which happens when a Dirac electron leaks into the region of high potential without any damping. The effect was originally proposed by Klein in 1930 for (real) relativistic electrons. But just after the discovery of graphene people are able to set up an experiment to verify the phenomenon for quasi relativistic particles, i.e the Dirac electron in graphene.[4]

We are particularly interested in transport properties of graphene. As described in [5], the conductivity of graphene shows a linear dependence on the chemical potential except very close to the neutrality point (where the Fermi level is exactly at the Dirac point). At the Dirac point, the conductivity reaches a minimum, which has the value of $\approx e^2/h$. Efforts have been put on explanation of this value. The expected value predicted from most theories is of the order of $e^2/\pi h$. Deviation of theoretical conductivity from the experiment inspired people to look at different mechanisms. Among them, electron-electron interaction is of interest.[6]

Recently, experiments are able to grow the high quality, suspended mono-layers graphene, which drive more attentions to electron-electron interaction.[7] In such a highly pure system, the electron-electron interaction becomes important. Studying the effect of interaction between the carriers (usually, of course, Coulomb interaction) leads to the series [9, 10, 11, 12], the general philosophy of which we also follow in this report.

The electron-electron interactions in graphene behave differently from normal doped semiconductors. To see the special nature of the electron-electron interactions in the case of graphene, we follow the simple argument of D. E. Sheehy and J. Schmalian.[8] Let us just have a look at the

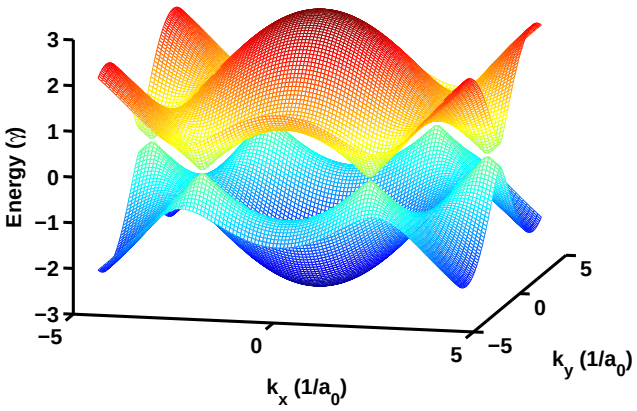


Figure 1: Simple tight-binding band structure of graphene. t is the hooping energy between nearest sites, a_0 is the lattice constants of graphene.

dimension of the ratio between the interaction energy and the kinetic energy of electrons in normal semi-conductors with quadratic dispersal relation,

$$[R_{\text{semi.}}] = \left[\frac{E_{\text{int.}}}{E_{\text{K.}}}\right]_{\text{semi.}} = \left[\frac{e^2/r_{ij}}{p^2/2m}\right] \propto \mathrm{L}^1, \tag{1}$$

which up to a physical constant, depends on dimension of length as L^1. This implies that, at low density the electron-electron interaction dominates. On the contrary, the kinetic energy is important at high density. But if we look at the same quantity for graphene,

$$[R_{\text{gr.}}] = \left[\frac{E_{\text{int.}}}{E_{\text{K.}}}\right]_{\text{gr.}} = \left[\frac{e^2/r_{ij}}{v_F p}\right] \propto \mathrm{L}^0, \tag{2}$$

one can see that electron-electron interaction and kinetic energy are equally important at any density.

Based on a general analysis of electron-electron interaction and the Boltzmann's kinetic equation formalism, L. Fritz *et al.* concentrated on the conductivity at the Dirac point.[11] The dc conductivity of pure graphene due to collision admits the "almost universal value" (at the Dirac point),

$$\sigma(\omega = 0) = 0.760\frac{e^2}{h\alpha^2}. \tag{3}$$

One can notice immediately that the result holds only for a system at exactly the Dirac point. Indeed, away from the neutrality point, the net force due to the external field on the charge carriers is non zero because of the non-zero total charge of the electron-hole gas, which results in an infinite conductivity (due to momentum conservation in the absence of impurity and neglecting the Umklapp processes). We will see that if we consider spin transport instead of electrical transport, we can extend this purely collision-limited transport theory to the regime away from the Dirac point.

To induce a spin current, similarly as the above study of electrical transport, we consider a graphene sheet in magnetic field, which is perpendicular to the sheet but inhomogeneous over the plane (with constant gradient). The force on the carriers now is generated by the gradient of the field coupling with the spins (neglecting orbital effect). In compare to the case of electrical transport, instead of the positive and negative charges, we are dealing with up and down spins. Now the net force on the system does not depend on the total charge but the total polarization of the system. So for the Fermi level far from the Dirac point, if we keep the system in zero polarization (zero net magnetic field) we will not obtain the infinite response but a finite, purely collision-controlled, that is, disorder-independent, result.

Due to the fact that spin conductivity is finite at finite chemical potential, spin transport becomes a candidate for studying the effect of electron-electron interaction in collision-dominated regime. Recently, experimentalists have been interested in spin transport in graphene.[13, 14] Our predictions of collision-limited spin diffusion here may hopefully be tested in ultra-clean, suspended graphene where impurities are sparse and the interactions are unscreened and hence strong.

Before going into details, let us summarize our results here. We will show that, surprisingly, dimensionless spin conductivity at the Dirac point is identical to the dimensionless electrical conductivity to the leading order in the interaction strength in a weak coupling approach. Effectively, at the Dirac point, an electron scatter from an other as if it "sees" only electrons of the same spin. Moreover, each projection of spins gives the same contribution to the spin current. Now, one just has to look at only one projection of spin, and one find exactly the same problem of electrical transport, where the electrical current is induced by an electric field.

In addition, analysis of the solution of the kinetic equation is investigated mainly via collinear limit approximation. This implies that scatterings in the same line are the strongest fact that courses the system tend to the stationary state. The collinear limit approximation was systematically studied in [9, 10, 11, 12] and others in the line.

Our estimation for the spin conductivity at and faraway from the Dirac point is found to be

$$\sigma_S = \begin{cases} \left(\frac{2\ln 2}{\pi}\right)^2 \frac{1}{C_1}\left[\frac{2\pi}{h}\right] & \text{for} \quad \mu = 0, \\ \left(\frac{\mu}{\pi T}\right)^2 \frac{1}{C_2}\left[\frac{2\pi}{h}\right] & \text{for} \quad \mu/T \gg 1, \end{cases} \tag{4}$$

where at the Dirac point $C_1 \approx 1.552\alpha^2$, and at large chemical potential limit the spin conductivity quadratically depends on reduced chemical potential μ/T with the asymptotic coefficient $C_2 \approx 9.0 \div 10.0(\alpha^2)$. Note that the unit for spin conductivity is $2\pi/h$, which is similar to the unit for electrical conductivity e^2/h, when spin plays the role of charge (up to the factor 2π).

The details of the numerical calculations are discussed in the last subsection of Section 2.

2 The kinetic equation for spin diffusion on graphene sheet

We will set out the kinetic formalism in the subsection 2.1. In the subsection 2.2 we show how to verify the equality between conductivity and spin conductivity by mapping between different scattering processes. The two last subsections, i.e subsection 2.3 and 2.4, are devoted to the collinear limit analysis and the discussion of the numerical implementation of the solution of the kinetic equation.

2.1 The Boltzmann kinetic formalism

We begin with the Hamiltonian for the system in second quantization language. The Hamiltonian for free Dirac electrons in graphene has the simple form,

$$H_0 = \sum_i \int \mathrm{d}\vec{r}[v_F \psi_i^\dagger (\vec{\sigma}\cdot\vec{p})\psi_i], \tag{5}$$

where v_F is the Fermi velocity at the Dirac point. The index i stands for the *"flavors of fermions"*, explicitly $\{i\} = \{\alpha, \sigma\}$, where α is *valley-index* (K or K') and σ is *spin-index* ($\uparrow$ or $\downarrow$). The Hamiltonian can be diagonalized by the Fourier transformation to momentum space,

$$\psi_i(\vec{r}, t) = \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2} C_{i\vec{k}}(t) e^{i\vec{k}\cdot\vec{r}}, \tag{6}$$

and then followed by a unitary transformation to diagonalize the *pseudo-spinor*,

$$\begin{pmatrix} C_{i1\vec{k}} \\ C_{i2\vec{k}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \frac{K}{|K|} & -\frac{K}{|K|} \end{pmatrix} \begin{pmatrix} \gamma_{i+\vec{k}} \\ \gamma_{i-\vec{k}} \end{pmatrix}, \tag{7}$$

where $K = k_x + ik_y$. In the equation, one can appreciate the presence of the indices $\pm$ as the *signs of energy* of electron (or we might call it *sign* for brevity). After such transformations, the Hamiltonian is diagonal,

$$H_0 = \sum_{i\lambda} \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2} \lambda \hbar v_F k \gamma^\dagger_{i\lambda\vec{k}} \gamma_{i\lambda\vec{k}}. \tag{8}$$

The Coulomb interaction term can be written in momentum presentation as

$$H_1 = \frac{1}{2} \sum_{ij} \int \frac{\mathrm{d}\vec{k}_1}{(2\pi)^2} \frac{\mathrm{d}\vec{k}_2}{(2\pi)^2} \frac{\mathrm{d}\vec{q}}{(2\pi)^2} C^\dagger_{i\vec{k}_1+\vec{q}} C^\dagger_{j\vec{k}_2-\vec{q}} V(q) C_{j\vec{k}_2} C_{i\vec{k}_1}, \tag{9}$$

where $V(q) = \hbar v_F \frac{2\pi\alpha}{q}$ is the Fourier transformation of the Coulomb potential. The Coulomb interaction strength is characterized by the "fine structure constant", $\alpha = \frac{1}{\epsilon}\frac{e^2}{\hbar v_F}$. The dielectric constant ϵ reflects the presence of substrates (up and down) on which the graphene sheet is deposited. In the case of pure, suspended graphene, we should have $\epsilon = 1$.[11]

Note that, renormalization group analysis of Coulomb interaction in graphene is discussed by González and other authors, which showed that Coulomb electron-electron interaction in graphene is of marginal irrelevant.[3, 6] Follow [11], our calculation is actually set up within the framework of renormalized theory. Namely, the coupling constant of interaction, the "fine structure constant" of graphene,

$$\alpha_0 = \frac{e^2}{\epsilon \hbar v_F}, \tag{10}$$

is renormalized as

$$\alpha(T) = \frac{\alpha_0}{1 + \alpha_0/4 \ln(\Lambda/T)}, \tag{11}$$

where Λ is the energy cut-off, which is of the order of the band width.

Upon applying the unitary transformation (7), the interaction term H_1 in the γ-presentation will be

$$H_1 = \sum_{ij} \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} \int \frac{\mathrm{d}\vec{k}_1}{(2\pi)^2} \frac{\mathrm{d}\vec{k}_2}{(2\pi)^2} \frac{\mathrm{d}\vec{q}}{(2\pi)^2} T_{\lambda_1\lambda_2\lambda_3\lambda_4}(\vec{k}_1, \vec{k}_2, \vec{q}) \gamma^\dagger_{i\lambda_4\vec{k}_1+\vec{q}} \gamma^\dagger_{j\lambda_3\vec{k}_2-\vec{q}} \gamma_{j\lambda_2\vec{k}_2} \gamma_{i\lambda_1\vec{k}_1}, \tag{12}$$

where

$$T_{\lambda_1\lambda_2\lambda_3\lambda_4}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) = \frac{V(q)}{8} \left[1 + \lambda_4\lambda_1 \frac{K_4^*}{|K_4|} \frac{K_1}{|K_1|}\right] \left[1 + \lambda_3\lambda_2 \frac{K_3^*}{|K_3|} \frac{K_2}{|K_2|}\right]. \tag{13}$$

Usually here and in the following we will use the notations $\vec{k}_3 = \vec{k}_2 - \vec{q}$ and $\vec{k}_4 = \vec{k}_1 + \vec{q}$ to simplify the expressions.

To reach our final goal of calculating the spin conductivity, we need the expression for the spin current operator,

$$\vec{j}_S = \sum_{\alpha\sigma\lambda} \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2} \sigma\lambda v_F \frac{\vec{k}}{k} \gamma^\dagger_{\alpha\sigma\lambda\vec{k}} \gamma_{\alpha\sigma\lambda\vec{k}}. \tag{14}$$

To deal with the effect of electron-electron scattering on the spin transport process in graphene, we use the Boltzmann kinetic equation, which is reviewed carefully in [15].

The distribution function is defined as the average of number operator in the γ-presentation,

$$f_{\alpha\sigma\lambda\vec{k}} = \langle \gamma^\dagger_{\alpha\sigma\lambda\vec{k}} \gamma_{\alpha\sigma\lambda\vec{k}} \rangle. \tag{15}$$

Note that we have the symmetry between the two valleys in the band-structure of graphene. As a result, distribution function will not depend on the valley-indices and we will omit these indices, except when explicitly indicated.

In equilibrium and without any magnetic field, the distribution does not depend on the projections of spin and is given by the Fermi-Dirac distribution,

$$f^0_{\lambda k} = \frac{1}{e^{(\lambda\hbar v_F k - \mu)/T} + 1}, \tag{16}$$

where μ is the chemical potential. At the Dirac point, $\mu = 0$, it takes the simple form,

$$f^0_{\lambda k} = \frac{1}{e^{\lambda\hbar v_F k/T} + 1}. \tag{17}$$

Now imagine we apply a magnetic field on the graphene sheet, which is along Oz direction and inhomogeneous over the plane Oxy. In such a field, spins up and down will experience forces in opposite directions due to the gradient of the magnetic field in the Oxy plane. The force will drive the system out of equilibrium. On the other hand, spins up and spins down moving in the opposite directions will collide with each other. The collisions relax system towards equilibrium. If the equilibrium number of spins up and spins down is equal, the system can always tend to a stationary state, where we have the exact compensation between the driving and the relaxation. On the contrary, if the system is polarized, it is easy to see that the total force acting on the system is non-zero and the force would make the total momentum keep increasing. In that case, there is no stationary state due to electron-electron scattering relaxation and an other relaxation mechanism should be invoked to obtain a finite response. In the following, we will always restrict ourselves to non-polarized systems, i.e systems at zero global field intensity.

Let us have a look at the distribution function of the perturbed system. In general, the change of distribution function is a summation of two parts, the external *field* and *scattering* terms,[1]

$$\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t} = \left[\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t}\right]_{\text{field}} + \left[\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t}\right]_{\text{scatt.}}. \tag{18}$$

In the stationary state, the distribution function will not depend on time, or in other words the change due to the external field is compensated by the scattering,

$$\left[\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t}\right]_{\text{field}} = -\left[\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t}\right]_{\text{scatt.}}. \tag{19}$$

By a standard procedure, the *driving term* due to external field can be found easily,

$$\left[\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t}\right]_{\text{field}} = -\dot{\vec{k}}\vec{\nabla}_{\vec{k}} f_{\sigma\lambda\vec{k}}, \tag{20}$$

where $\dot{\vec{k}} = \delta\vec{k}/\delta t$ is the changing rate of Bloch wave vector upon applying the external field, which can be found by the Newtonian-like equation, $\hbar\delta\vec{k} = \vec{F}\delta t$, where $\vec{F}$ is the force acting on the spins in inhomogeneous magnetic field.[15] If magnetic field is along the z axis, we can easily find that $\vec{F} = \sigma\mu_B\vec{\nabla}_{(x,y)}B_z$ where μ_B is Bohr magneton (with minus sign of e) and the nabla operator is taken on the plane Oxy. Putting the rate into the Eq. (20), one finds

$$\left[\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t}\right]_{\text{field}} = -\sigma\frac{\vec{F}_0}{\hbar}\cdot\vec{\nabla}_{\vec{k}} f_{\sigma\lambda\vec{k}}, \tag{21}$$

where $\vec{F}_0 = \mu_B\vec{\nabla}_{(x,y)}B_z$.

We suppose that the external field is small enough so that we can treat it in the first order of perturbation theory (linear response). In order to use perturbation theory, we change $\vec{F}_0 \to z\vec{F}_0$, where z is used to keep track of the order of perturbation. Accordingly, the distribution function will change by a small difference $f_{\sigma\lambda\vec{k}} = f^0_{\lambda k} + z g_{\sigma\lambda\vec{k}} f^0_{\lambda k}[1 - f^0_{\lambda k}]$, where f^0 is the equilibrium distribution. We will call g the *reduced distribution function*.[2] Inserting the expression of the distribution function into Eq. (21) and keeping the first order of z we obtain the linearized driving term,

$$\left[\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t}\right]_{\text{field}} = \frac{\sigma\lambda v_F}{T}\vec{F}_0\frac{\vec{k}}{k} f^0_{\lambda k}[1 - f^0_{\lambda k}]. \tag{22}$$

We now turn to studying the scattering term in the right-hand side of the Boltzmann equation (19). In [11], the authors have derived the scattering term by the Green function method. In our case here, we restrict ourselves to semi-rigorous arguments as described in [15], which are known to be equivalent to the Green's function method. In general, the scattering term has the form

$$\left[\frac{\mathrm{d}f_1}{\mathrm{d}t}\right]_{\text{scatt.}} = -\sum_{2,3,4} Q(1,2;3,4)\{f_1 f_2[1 - f_3][1 - f_4] - [1 - f_1][1 - f_2] f_3 f_4\}, \tag{23}$$

[1] In our case, temperature is assumed to be homogeneous over the graphene sheet, i.e we will not consider the diffusion process due to the gradient of temperature.

[2] For systems in a zero global field, the *reduced distribution function* is always anti symmetric with respect to the two signs of σ, $g_{\sigma\lambda\vec{k}} = \sigma g_{\lambda\vec{k}}$. For the purpose of keeping a general form, we usually do not explicitly include this dependence in the formula, but it is always implicit. .

where as a shorthand notation, we used the collective indices $\{r\} = \{\sigma_r, \lambda_r, \vec{k}_r\}$, with r runs over $\{1,2,3,4\}$. The term $Q(1,2;3,4)$ stands for the transition rate from state $(1,2)$ to state $(3,4)$. The term $F(1,2,\overline{3},\overline{4}) = f_1 f_2[1-f_3][1-f_4]$ is the probability that states $(1,2)$ is occupied and states $(3,4)$ is empty. A similar interpretation applies to $F(\overline{1},\overline{2},3,4) = [1-f_1][1-f_2]f_3 f_4$. The summation is taken over all the states $(2,3,4)$ (integrals are to be understood for continuous indices). We also want to notice here the symmetry property of the transition term, $Q(1,2;3,4) = Q(2,1;3,4) = Q(1,2;4,3) = Q(3,4;1,2)$ due to the principle of *identical particles* and *microscopic reversibility*, as discussed in Chapter 7 of [15].

Let us have a look at the probability term $F(1,2,3,4) = F(1,2,\overline{3},\overline{4}) - F(\overline{1},\overline{2},3,4)$. As we have done with the driving term, we wish to linearize the probability term. Inserting $f = f^0 + zgf^0[1-f^0]$ into the expression and keeping the first order of z, we can factor out the expression,

$$F(1,2,3,4) \approx F^0(1,2,3,4)G(1,2,3,4), \tag{24}$$

where the constant part $F^0(1,2,3,4) = f_1^0 f_2^0[1-f_3^0][1-f_4^0] = [1-f_1^0][1-f_2^0]f_3^0 f_4^0$ is called *main factor* and the linear part $G(1,2,3,4) = g_1 + g_2 - g_3 - g_4$ is called *reduced density* (of transition).

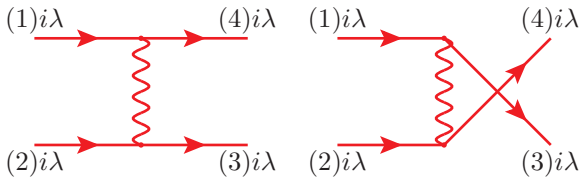
Now we concentrate on the transition parts. Note that due the Fermi's golden rule,[16]

$$W_{i\to f} = \frac{2\pi}{\hbar}|\langle f|H_1|i\rangle|^2\delta(E_f - E_i), \tag{25}$$

the transition rates always contain the factor $2\pi/\hbar$ and the δ-function standing for energy conservation and the square of matrix elements. For simplicity, in following we also use the name *transition rates* for just the square of matrix elements, and the energy conservation will be inserted in the last expressions.

Due to the values of different discrete indices (i.e flavors and signs) in scattering process, the transition rates can be divided in five classes described by the corresponding diagrams below.

1. Scattering between particles with the same sign of energy and the same flavor (let us remind that "flavor" i is the collective index for valley-index α and spin-index σ, $i = \{\alpha, \sigma\}$):

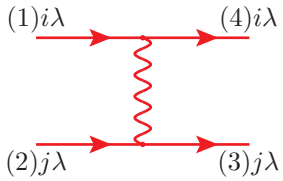


$$R_1(\vec{k}_1,\vec{k}_2,\vec{q}) = 2\left|T_{++++}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) - T_{++++}(\vec{k}_1,\vec{k}_2,\vec{k}_4,\vec{k}_3)\right|^2. \tag{26}$$

The corresponding reduced density for the transition is

$$\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\lambda\vec{k}_2,\sigma\lambda\vec{k}_3,\sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\sigma\lambda\vec{k}_2} - g_{\sigma\lambda\vec{k}_3} - g_{\sigma\lambda\vec{k}_4}. \tag{27}$$

2. Scattering between particles with the same sign but different flavors:



$$R_2(\vec{k_1},\vec{k}_2,\vec{q}) = 4\left|T_{++++}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\right|^2. \tag{28}$$

This again is divided in two sub-classes (two kinds of different flavors): the same spin but different valleys, different spins (no matter which valleys they are in, this gives the factor of 2 to the corresponding reduced density in the Boltzmann equation written below). The rate is the same for the two sub-classes, but the reduced densities will be different,

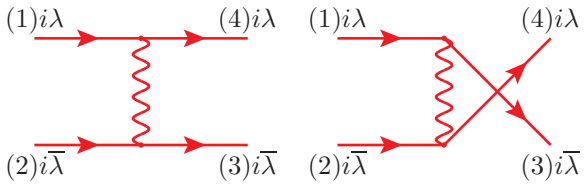
$$\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\lambda\vec{k}_2,\sigma\lambda\vec{k}_3,\sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\sigma\lambda\vec{k}_2} - g_{\sigma\lambda\vec{k}_3} - g_{\sigma\lambda\vec{k}_4} \tag{29}$$

for the former, and

$$\mathcal{G}(\sigma\lambda\vec{k}_1,\overline{\sigma}\lambda\vec{k}_2,\overline{\sigma}\lambda\vec{k}_3,\sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\overline{\sigma}\lambda\vec{k}_2} - g_{\overline{\sigma}\lambda\vec{k}_3} - g_{\sigma\lambda\vec{k}_4} \tag{30}$$

for the latter. For simplicity, we use over-bar notation to indicate the minus sign, i.e $\overline{\sigma} = -\sigma$ and $\overline{\lambda} = -\lambda$.

3. Scattering between particles with different signs but the same flavor:

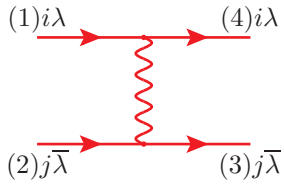


$$R_3(\vec{k}_1,\vec{k}_2,\vec{q}) = 4\left|T_{+--+}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) - T_{+-+-}(\vec{k}_1,\vec{k}_2,\vec{k}_4,\vec{k}_3)\right|^2. \tag{31}$$

The reduced density is

$$\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\sigma\overline{\lambda}\vec{k}_2} - g_{\sigma\overline{\lambda}\vec{k}_3} - g_{\sigma\lambda\vec{k}_4}. \tag{32}$$

4. Scattering between particles with different signs of energy and different flavors:



$$R_4(\vec{k}_1,\vec{k}_2,\vec{q}) = 4\left|T_{+--+}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\right|^2. \tag{33}$$

Like the case 1, we have to divide the processes into two sub-classes. The reduced densities are

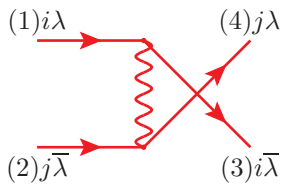
$$\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\sigma\overline{\lambda}\vec{k}_2} - g_{\sigma\overline{\lambda}\vec{k}_3} - g_{\sigma\lambda\vec{k}_4} \tag{34}$$

for particles of the same spin, and

$$\mathcal{G}(\sigma\lambda\vec{k}_1,\overline{\sigma}\overline{\lambda}\vec{k}_2,\overline{\sigma}\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\overline{\sigma}\overline{\lambda}\vec{k}_2} - g_{\overline{\sigma}\overline{\lambda}\vec{k}_3} - g_{\sigma\lambda\vec{k}_4} \tag{35}$$

for particles of different spins.

5. Scattering between particles with different signs of energy, different flavors with signs exchanging:



$$R_5(\vec{k}_1,\vec{k}_2,\vec{q}) = 4\left|T_{+-+-}(\vec{k}_1,\vec{k}_2,\vec{k}_4,\vec{k}_3)\right|^2. \tag{36}$$

The corresponding reduced densities are

$$\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\sigma\overline{\lambda}\vec{k}_2} - g_{\sigma\overline{\lambda}\vec{k}_3} - g_{\sigma\lambda\vec{k}_4} \tag{37}$$

for particles of the same spin, and

$$\mathcal{G}(\sigma\lambda\vec{k}_1,\overline{\sigma}\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\overline{\sigma}\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\overline{\sigma}\overline{\lambda}\vec{k}_2} - g_{\sigma\overline{\lambda}\vec{k}_3} - g_{\overline{\sigma}\lambda\vec{k}_4} \tag{38}$$

for particles of different spins.

For more details of deriving the expressions for the scattering rates, we refer to Appendix A. Note that the momentum conservation is satisfied automatically when we explicitly insert $\vec{k}_3 = \vec{k}_2 - \vec{q}$ and $\vec{k}_4 = \vec{k}_1 + \vec{q}$.

Putting all the densities and transition rates together (and insert the energy conservation) we have the final form of the collision integral,

$$\left[\frac{\mathrm{d}f_1}{\mathrm{d}t}\right]_{\text{scatt.}} = -\frac{2\pi}{\hbar}\frac{1}{\hbar v_F}\int\frac{\mathrm{d}\vec{k}_2}{(2\pi)^2}\frac{\mathrm{d}\vec{q}}{(2\pi)^2}[Q_1(\vec{k}_1,\vec{k}_2,\vec{q}) + Q_2(\vec{k}_1,\vec{k}_2,\vec{q})] \tag{39}$$

Where $Q_1(\vec{k}_1,\vec{k}_2,\vec{q})$ stands for collisions between the particles of the same sign of energy, which is

$$\begin{aligned} Q_1(\vec{k}_1,\vec{k}_2,\vec{q}) &= \delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1,\lambda k_2,\lambda k_3,\lambda k_4) \\ &\times\Big\{\ R_1(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\lambda\vec{k}_2,\sigma\lambda\vec{k}_3,\sigma\lambda\vec{k}_4)] \\ &\quad + R_2(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\lambda\vec{k}_2,\sigma\lambda\vec{k}_3,\sigma\lambda\vec{k}_4) \\ &\qquad + 2\mathcal{G}(\sigma\lambda\vec{k}_1,\overline{\sigma}\lambda\vec{k}_2,\overline{\sigma}\lambda\vec{k}_3,\sigma\lambda\vec{k}_4)]\Big\}, \end{aligned} \tag{40}$$

and $Q_2(\vec{k}_1,\vec{k}_2,\vec{q})$ stands for collisions between the particles of different signs of energy, which is

$$\begin{aligned} Q_2(\vec{k}_1,\vec{k}_2,\vec{q}) &= \delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1,\overline{\lambda}k_2,\overline{\lambda}k_3,\lambda k_4) \\ &\times\Big\{\ R_3(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4)] \\ &\quad + R_4(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4) \\ &\qquad + 2\mathcal{G}(\sigma\lambda\vec{k}_1,\overline{\sigma}\overline{\lambda}\vec{k}_2,\overline{\sigma}\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4)] \\ &\quad + R_5(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{G}(\sigma\lambda\vec{k}_1,\sigma\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4) \\ &\qquad + 2\mathcal{G}(\sigma\lambda\vec{k}_1,\overline{\sigma}\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\overline{\sigma}\lambda\vec{k}_4)]\Big\}. \end{aligned} \tag{41}$$

The *main factor* of density is

$$F^0(\lambda_1 k_1, \lambda_2 k_2, \lambda_3 k_3, \lambda_4 k_4) = \frac{1}{e^{+(\lambda_1 \hbar v_F k_1 - \mu)/T}+1}\frac{1}{e^{+(\lambda_2 \hbar v_F k_2 - \mu)/T}+1} \times \frac{1}{e^{-(\lambda_3 \hbar v_F k_3 - \mu)/T}+1}\frac{1}{e^{-(\lambda_4 \hbar v_F k_4 - \mu)/T}+1}, \quad (42)$$

and the *reduced density* is

$$\mathcal{G}(\sigma_1\lambda_1\vec{k}_1, \sigma_2\lambda_2\vec{k}_2, \sigma_3\lambda_3\vec{k}_3, \sigma_4\lambda_4\vec{k}_4) = g_{\sigma_1\lambda_1\vec{k}_1} + g_{\sigma_2\lambda_2\vec{k}_2} - g_{\sigma_3\lambda_3\vec{k}_3} - g_{\sigma_4\lambda_4\vec{k}_4}, \quad (43)$$

with $\vec{k}_3 = \vec{k}_2 - \vec{q}$ and $\vec{k}_4 = \vec{k}_1 + \vec{q}$.

We can consider the collision integral as a linear operator $\hat{C}$ acting on the function $g_{\sigma\lambda\vec{k}}$,

$$\left[\frac{\mathrm{d}f_{\sigma\lambda\vec{k}}}{\mathrm{d}t}\right]_{\text{scatt.}} = -\hat{C}g_{\sigma\lambda\vec{k}}, \quad (44)$$

where the minus sign is kept to make the *collision operator* positive defined.

The Boltzmann's equation now is simply

$$\hat{C}g = D, \quad (45)$$

where D is the *driving-term* (22).

With the solution of the Boltzmann equation at hand, we can calculate the spin current with the aid of Eq. (14) as

$$\begin{aligned}\vec{j}_S &= 2v_F\sum_{\sigma\lambda}\int\frac{\mathrm{d}\vec{k}}{(2\pi)^2}\sigma\lambda\frac{\vec{k}}{k}f_{\sigma\lambda\vec{k}} \\ &= 2v_F\sum_{\sigma\lambda}\int\frac{\mathrm{d}\vec{k}}{(2\pi)^2}\sigma\lambda\frac{\vec{k}}{k}[f_{\sigma\lambda\vec{k}} - f^0_{\lambda k}] \\ &= 2v_F\sum_{\sigma\lambda}\int\frac{\mathrm{d}\vec{k}}{(2\pi)^2}\sigma\lambda\frac{\vec{k}}{k}\frac{\vec{k}\vec{e}}{k}g_{\sigma\lambda k}f^0_{\lambda k}[1 - f^0_{\lambda k}], \end{aligned} \quad (46)$$

where we have used the fact that in equilibrium spin current is identically zero. Moreover, we restored the scalar form of the deviation g. Projecting the expression onto the axis of external force we can easily perform the integral over angle,

$$\begin{aligned} j_S &= \frac{2v_F}{(2\pi)^2}\sum_{\sigma\lambda}\int_0^{2\pi}\mathrm{d}\phi\cos^2\phi\int_0^{+\infty}k\mathrm{d}k\ \sigma\lambda g_{\sigma\lambda k}f^0_{\lambda k}[1 - f^0_{\lambda k}] \\ &= \frac{v_F}{2\pi}\sum_{\sigma\lambda}\int_0^{+\infty}k\mathrm{d}k\sigma\lambda g_{\sigma\lambda k}f^0_{\lambda k}[1 - f^0_{\lambda k}], \end{aligned} \quad (47)$$

where the factor 2 reflects valley degeneracy in graphene.

And then, the spin conductivity is defined as the spin current per external force unit (i.e F_0),

$$\sigma_S = \frac{j_S}{F_0}. \quad (48)$$

Returning to the collision operator, due to the symmetry of the transition rate (in general) noted above one can prove that $\hat{C}$ is *self-adjoint* and *positive defined* with respect to the inner-product

$$\langle h, g\rangle = \sum_{\sigma\lambda} \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2} h_{\sigma\lambda\vec{k}} g_{\sigma\lambda\vec{k}}. \tag{49}$$

The Boltzmann equation (45) can be written in variational form with the aid of the functional $\mathrm{F}[g]$ defined as

$$\mathrm{F}[g] = \frac{1}{2}\langle g, \hat{C}g\rangle - \langle g, D\rangle. \tag{50}$$

The solution of the Boltzmann equation is the minimum of the functional.[3] This variational form will be particularly useful for numerical treatment of the equation. Namely, to solve the equation, we will choose a set of basic functions $\{b_i\}_{i=1}^N$ for g, with the assumption that $g = \sum_{i=1}^N \chi_i b_i$ is a good approximation. Putting the approximation into the functional $\mathrm{F}[g]$ and minimizing the value of functional with respect to the coefficient $\{\chi_i\}$, actually we end up with a linear system of equations for $\{\chi_i\}$,

$$\sum_{j=1}^{N} C_{ij}\chi_j = D_i, \tag{52}$$

where the "matrix elements" are $C_{ij} = \langle b_i, \hat{C}b_j\rangle$ and $D_i = \langle b_i, D\rangle$.

We also notice that the matrix element of the form $\langle h, \hat{C}g\rangle$ can be written in a more symmetric way using the symmetry property of transition rates,

$$\langle h, \hat{C}g\rangle = \frac{2\pi}{\hbar}\frac{1}{\hbar v_F}\sum_{\sigma\lambda}\int \frac{\mathrm{d}\vec{k}_1}{(2\pi)^2}\frac{\mathrm{d}\vec{k}_2}{(2\pi)^2}\frac{\mathrm{d}\vec{q}}{(2\pi)^2}[QF_1(\vec{k}_1,\vec{k}_2,\vec{q}) + QF_2(\vec{k}_1,\vec{k}_2,\vec{q})], \tag{53}$$

where, again, $QF_1(\vec{k}_1,\vec{k}_2,\vec{q})$ stands for collisions between the particles of the same sign of energy, which is

$$\begin{aligned} QF_1(\vec{k}_1,\vec{k}_2,\vec{q}) &= \delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1, \lambda k_2, \lambda k_3, \lambda k_4) \\ &\times \Big\{ R_1(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{GF}(\sigma\lambda\vec{k}_1, \sigma\lambda\vec{k}_2, \sigma\lambda\vec{k}_3, \sigma\lambda\vec{k}_4)] \\ &\quad + R_2(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{GF}(\sigma\lambda\vec{k}_1, \sigma\lambda\vec{k}_2, \sigma\lambda\vec{k}_3, \sigma\lambda\vec{k}_4) \\ &\qquad + 2\mathcal{GF}(\sigma\lambda\vec{k}_1, \overline{\sigma}\lambda\vec{k}_2, \overline{\sigma}\lambda\vec{k}_3, \sigma\lambda\vec{k}_4)]\Big\}, \end{aligned} \tag{54}$$

and $QF_2(\vec{k}_1,\vec{k}_2,\vec{q})$ stands for collisions between the particles of different signs of energy, which is

$$\begin{aligned} QF_2(\vec{k}_1,\vec{k}_2,\vec{q}) &= \delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1, \overline{\lambda} k_2, \overline{\lambda} k_3, \lambda k_4) \\ &\times \Big\{ R_3(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{GF}(\sigma\lambda\vec{k}_1, \sigma\overline{\lambda}\vec{k}_2, \sigma\overline{\lambda}\vec{k}_3, \sigma\lambda\vec{k}_4)] \end{aligned}$$

[3] Using the same line described in [15], one can also proof an alternative form of the variational principle,

$$\sigma_S = \max_g \left\{ \frac{2T}{F_0^2}\frac{\langle g, D\rangle^2}{\langle g, \hat{C}g\rangle} \right\}. \tag{51}$$

In that form, one can see the direct appearance of spin conductivity involving.

$$
\begin{aligned}
&+R_4(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{GF}(\sigma\lambda\vec{k}_1,\sigma\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4)\\
&\qquad +2\mathcal{GF}(\sigma\lambda\vec{k}_1,\overline{\sigma}\overline{\lambda}\vec{k}_2,\overline{\sigma}\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4)]\\
&+R_5(\vec{k}_1,\vec{k}_2,\vec{q})[\mathcal{GF}(\sigma\lambda\vec{k}_1,\sigma\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\sigma\lambda\vec{k}_4)\\
&\qquad +2\mathcal{GF}(\sigma\lambda\vec{k}_1,\overline{\sigma}\overline{\lambda}\vec{k}_2,\sigma\overline{\lambda}\vec{k}_3,\overline{\sigma}\lambda\vec{k}_4)]\Big\}, \qquad (55)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{GF}(\sigma_1\lambda_1\vec{k}_1,\sigma_2\lambda_2\vec{k}_2,\sigma_3\lambda_3\vec{k}_3,\sigma_4\lambda_4\vec{k}_4) = \;& \tfrac{1}{4}\,[h_{\sigma_1\lambda_1\vec{k}_1}+h_{\sigma_2\lambda_2\vec{k}_2}-h_{\sigma_3\lambda_3\vec{k}_3}-h_{\sigma_4\lambda_4\vec{k}_4}]\\
&\times[g_{\sigma_1\lambda_1\vec{k}_1}+g_{\sigma_2\lambda_2\vec{k}_2}-g_{\sigma_3\lambda_3\vec{k}_3}-g_{\sigma_4\lambda_4\vec{k}_4}]. \qquad (56)
\end{aligned}
$$

And moreover, we notice that, because the driving term $D=\frac{\sigma\lambda v_F}{T}\vec{F}_0\frac{\vec{k}}{k}f^0_{\lambda k}[1-f^0_{\lambda k}]$ depends on directions as the dot-product of $\frac{\vec{k}}{k}$ and the direction of external force $\vec{e}=\frac{\vec{F}_0}{F_0}$, we expect the same form of dependence on direction of the solution g, i.e $g_{\sigma\lambda\vec{k}}=\vec{e}\frac{\vec{k}}{k}g_{\sigma\lambda k}$. Putting this form into the equation, and removing the factor $\vec{e}$ on both sides of the equation we will have a vector equation, where the driving term, $D=\frac{\sigma\lambda v_F}{T}F_0\frac{\vec{k}}{k}f^0_{\lambda k}[1-f^0_{\lambda k}]$, and the reduced distribution function, $g_{\sigma\lambda\vec{k}}=\frac{\vec{k}}{k}g_{\sigma\lambda k}$, can be understood as vector-functions. The arguments presented above will essentially remain the same, as long as the product between numbers in the definition of inner product of h and g should be replaced by dot-product between vectors, and so on. In particular, the Eq. (56) will change to

$$
\begin{aligned}
\mathcal{GF}(\sigma_1\lambda_1\vec{k}_1,\sigma_2\lambda_2\vec{k}_2,\sigma_3\lambda_3\vec{k}_3,\sigma_4\lambda_4\vec{k}_4) = \;& \tfrac{1}{4}[\tfrac{\vec{k}_1}{k_1}h_{\sigma_1\lambda_1k_1}+\tfrac{\vec{k}_2}{k_2}h_{\sigma_2\lambda_2k_2}-\tfrac{\vec{k}_3}{k_3}h_{\sigma_3\lambda_3k_3}-\tfrac{\vec{k}_4}{k_4}h_{\sigma_4\lambda_4k_4}]\\
&\cdot[\tfrac{\vec{k}_1}{k_1}g_{\sigma_1\lambda_1k_1}+\tfrac{\vec{k}_2}{k_2}g_{\sigma_2\lambda_2k_2}-\tfrac{\vec{k}_3}{k_3}g_{\sigma_3\lambda_3k_3}-\tfrac{\vec{k}_4}{k_4}g_{\sigma_4\lambda_4k_4}]. \qquad (57)
\end{aligned}
$$

Before coming to analyze the solution of the kinetic equation, we rescale the unit for collision operator and also the driving term. The unit to measure energy is naturally T. This immediately gives a natural unit for wave vectors, $T/\hbar v_F$. As a result, we can work with dimensionless quantities by setting $\hbar=1$, $v_F=1$, $T=1$. Unless indicated explicitly, we will always use the dimensionless units system in following sections.

By rescaling the unit for our problem, one can immediately see that the only relevant parameter is μ/T. As a result, the spin conductivity, apart from the unit of $2\pi/h$, is solely a function of μ/T. In the next subsections, we will study in detail the function $\sigma_S(\mu/T)$, its minimum at $\mu/T=0$ and its asymptotic behavior at high chemical potential, $\mu/T\gg 1$.

2.2 Transport at the Dirac point: mapping spin transport into electrical transport

In this section we are going to show that to the lowest order in the interactions (Born approximation), it is possible to map the problem of spin transport into that of electrical transport at the Dirac point. In those conditions, the collision between particles of different spins can be mapped into collision between particles of the same spin. Effectively, only electrons of the same spin can "see" each other, or in other word, the dynamics of two projections of spin decouple from each other. If one now looks at only one projection of spin (*up* or *down*), one will find the problem of electrical transport, where electrons move under external force (F_0 or $-F_0$) and collide between them (*up* or *down*) at the same time, giving a finite current of particles.

First of all, we would like to notice the well-known fact that the dynamics of negative energy electrons can be mapped into dynamic of positive energy holes. Moreover, in the collision integral, one can map the terms of collisions between particles of different signs into collision between particles of the same sign. Indeed, the integral involving collisions between particles of the same sign is of the form

$$J_1 = \int \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{q} \delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|) \mathcal{F}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4). \tag{58}$$

Whereas the integral involving collisions between particles of different signs is of the form

$$J_2 = \int \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{q} \delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|) \mathcal{F}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4). \tag{59}$$

We, we remind that $\vec{k}_3 = \vec{k}_2 - \vec{q}$ and $\vec{k}_4 = \vec{k}_1 + \vec{q}$. Or, if we restore the δ-function due to momentum conservation in the expressions, the integrals would be

$$J_1 = \int \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{k}_3 \mathrm{d}\vec{k}_4 \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|) \mathcal{F}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4), \tag{60}$$

$$J_2 = \int \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{k}_3 \mathrm{d}\vec{k}_4 \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|) \mathcal{F}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4). \tag{61}$$

Now, it is easy to see that we can make J_2 have the same form as J_1 by simply changing the variables ($\vec{k}_2 \to -\vec{k}_3, \vec{k}_3 \to -\vec{k}_2$) and end up with

$$J_2 = \int \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{k}_3 \mathrm{d}\vec{k}_4 \delta(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4) \delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|) \mathcal{F}(\vec{k}_1, -\vec{k}_3, -\vec{k}_2, \vec{k}_4). \tag{62}$$

Using this transformation, we can show a quite interesting property of collision operator at the Dirac point. At the neutrality point, the solution is anti symmetric with respect to the two signs of σ, $g_{\sigma\lambda k} = \sigma\lambda g_k$. We are going to show that

$$\hat{C}[\sigma\lambda g_{\vec{k}}] = \sigma \hat{C}[\lambda g_{\vec{k}}]. \tag{63}$$

This equation needs some comments. On the left-hand side, we have the collision operator acting on the reduced distribution function, which is anti-symmetry with respect to the two variables σ, and λ. On the right-hand side, the collision operator acts on the reduced distribution, which is still anti-symmetric with respect to λ, but now is symmetric with respect to σ. To do that, using Eq. (62) we transform the terms involving R_2, R_4 in Eqs. (40) and (41) as follows,

$$\begin{aligned}
&\int \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{q} \delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|) F^0(\lambda k_1, \lambda k_2, \lambda k_3, \lambda k_4) \\
&\qquad R_2(\vec{k}_1, \vec{k}_2, \vec{q}) 2\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\lambda\vec{k}_2, \overline{\sigma}\lambda\vec{k}_3, \sigma\lambda\vec{k}_4) \\
= &\int \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 \mathrm{d}\vec{q} \delta(|\vec{k}_1| + |\vec{k}_3| - |\vec{k}_2| - |\vec{k}_4|) F^0(\lambda k_1, \lambda k_3, \lambda k_2, \lambda k_4) \\
&\qquad R_2(\vec{k}_1, -\vec{k}_2 + \vec{q}, \vec{q}) 2\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\lambda(-\vec{k}_3), \overline{\sigma}\lambda(-\vec{k}_2), \sigma\lambda\vec{k}_4).
\end{aligned} \tag{64}$$

One can easily prove that at the Dirac point $\mu = 0$, with the symmetry between two projections of spin and two signs of energy, we have

$$F^0(\lambda k_1, \lambda k_3, \lambda k_2, \lambda k_4) = F^0(\lambda k_1, \overline{\lambda} k_2, \overline{\lambda} k_3, \lambda k_4), \tag{65}$$

and also

$$\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\lambda(-\vec{k}_3), \overline{\sigma}\lambda(-\vec{k}_2), \sigma\lambda\vec{k}_4) = \mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\overline{\lambda}\vec{k}_2, \sigma\overline{\lambda}\vec{k}_3, \sigma\lambda\vec{k}_4). \tag{66}$$

More over, one can directly check that

$$R_2(\vec{k}_1, -\vec{k}_2 + \vec{q}, \vec{q}) = R_4(\vec{k}_1, \vec{k}_2, \vec{q}). \tag{67}$$

Then it follows that

$$\begin{aligned}
&\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{q}\delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1, \lambda k_2, \lambda k_3, \lambda k_4) \\
&\qquad \times R_2(\vec{k}_1, \vec{k}_2, \vec{q})2\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\lambda\vec{k}_2, \overline{\sigma}\lambda\vec{k}_3, \sigma\lambda\vec{k}_4) \\
=&\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{q}\delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1, \overline{\lambda} k_2, \overline{\lambda} k_3, \lambda k_4) \\
&\qquad \times R_4(\vec{k}_1, \vec{k}_2, \vec{q})2\mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\overline{\lambda}\vec{k}_2, \sigma\overline{\lambda}\vec{k}_3, \sigma\lambda\vec{k}_4).
\end{aligned} \tag{68}$$

Similarly, we can prove another equality,

$$\begin{aligned}
&\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{q}\delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1, \overline{\lambda} k_2, \overline{\lambda} k_3, \lambda k_4) \\
&\qquad \times R_4(\vec{k}_1, \vec{k}_2, \vec{q})2\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\overline{\lambda}\vec{k}_2, \overline{\sigma}\overline{\lambda}\vec{k}_3, \sigma\lambda\vec{k}_4) \\
=&\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{q}\delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1, \lambda k_2, \lambda k_3, \lambda k_4) \\
&\qquad \times R_2(\vec{k}_1, \vec{k}_2, \vec{q})2\mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\lambda\vec{k}_2, \sigma\lambda\vec{k}_3, \sigma\lambda\vec{k}_4),
\end{aligned} \tag{69}$$

and finally, we transform the last term involving R_5 with signs exchanging process by replacing $(\vec{k}_2 \to -\vec{k}_2, \vec{k}_4 \to -\vec{k}_4)$. Note that, δ-function, F_0, R_5 are unchanged under such a transformation, whereas $\mathcal{G}$ change the signs before $\vec{k}_2$ and $\vec{k}_4$,

$$\begin{aligned}
&\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{q}\delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1, \overline{\lambda} k_2, \overline{\lambda} k_3, \lambda k_4) \\
&\qquad \times R_5(\vec{k}_1, \vec{k}_2, \vec{q})2\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\overline{\lambda}\vec{k}_2, \sigma\overline{\lambda}\vec{k}_3, \overline{\sigma}\lambda\vec{k}_4) \\
=&\int \mathrm{d}\vec{k}_1\mathrm{d}\vec{k}_2\mathrm{d}\vec{q}\delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|)F^0(\lambda k_1, \overline{\lambda} k_2, \overline{\lambda} k_3, \lambda k_4) \\
&\qquad \times R_5(\vec{k}_1, \vec{k}_2, \vec{q})2\mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\overline{\lambda}\vec{k}_2, \sigma\overline{\lambda}\vec{k}_3, \sigma\lambda\vec{k}_4).
\end{aligned} \tag{70}$$

Again, note that for $g_{\sigma\lambda\vec{k}} = \sigma\lambda g_{\vec{k}}$, the term $\mathcal{G}$ is simply

$$\begin{aligned}
\mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\lambda\vec{k}_2, \sigma\lambda\vec{k}_3, \sigma\lambda\vec{k}_4) &= \sigma\lambda\left(g_{\vec{k}_1} + g_{\vec{k}_2} - g_{\vec{k}_3} - g_{\vec{k}_4}\right), \\
\mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\overline{\lambda}\vec{k}_2, \sigma\overline{\lambda}\vec{k}_3, \sigma\lambda\vec{k}_4) &= \sigma\lambda\left(g_{\vec{k}_1} - g_{\vec{k}_2} + g_{\vec{k}_3} - g_{\vec{k}_4}\right).
\end{aligned} \tag{71}$$

Now the operator can be rewritten as

$$\hat{C}[\sigma\lambda g_k] = 2\pi\int \frac{\mathrm{d}\vec{k}_1}{(2\pi)^2}\frac{\mathrm{d}\vec{k}_2}{(2\pi)^2}\frac{\mathrm{d}\vec{q}}{(2\pi)^2}[Q_1^0(\vec{k}_1, \vec{k}_2, \vec{q}) + Q_2^0(\vec{k}_1, \vec{k}_2, \vec{q})], \tag{72}$$

where, as before, $Q_1^0(\vec{k}_1, \vec{k}_2, \vec{q})$ stands for collisions between the particles of the same sign, which is

$$\begin{aligned} Q_1^0(\vec{k}_1, \vec{k}_2, \vec{q}) &= \delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|) \\ &\times F^0(\lambda k_1, \lambda k_2, \lambda k_3, \lambda k_4) \\ &\times \left[R_1(\vec{k}_1, \vec{k}_2, \vec{q}) + 3R_2(\vec{k}_1, \vec{k}_2, \vec{q})\right] \\ &\times \sigma\lambda \left(g_{\vec{k}_1} + g_{\vec{k}_2} - g_{\vec{k}_3} - g_{\vec{k}_4}\right), \end{aligned} \tag{73}$$

and $Q_2^0(\vec{k}_1, \vec{k}_2, \vec{q})$ stands for collisions between the particles of different signs, which is

$$\begin{aligned} Q_2^0(\vec{k}_1, \vec{k}_2, \vec{q}) &= \delta(|\vec{k}_1| - |\vec{k}_2| + |\vec{k}_3| - |\vec{k}_4|) \\ &\times F^0(\lambda k_1, \overline{\lambda} k_2, \overline{\lambda} k_3, \lambda k_4) \\ &\times \left[R_3(\vec{k}_1, \vec{k}_2, \vec{q}) + 3R_4(\vec{k}_1, \vec{k}_2, \vec{q}) + 3R_5(\vec{k}_1, \vec{k}_2, \vec{q})\right] \\ &\times \sigma\lambda \left(g_{\vec{k}_1} + g_{\vec{k}_2} - g_{\vec{k}_3} - g_{\vec{k}_4}\right). \end{aligned} \tag{74}$$

The right-hand side of Eq. (72) is nothing but $\sigma\hat{C}[\lambda g_{\vec{k}}]$ and the above equation just implies $\hat{C}[\sigma\lambda g_{\vec{k}}] = \sigma\hat{C}[\lambda g_{\vec{k}}]$.

If one notices that $F^0(\lambda k_1, \lambda k_2, \lambda k_3, \lambda k_4)$, and $F^0(\lambda k_1, \overline{\lambda} k_2, \overline{\lambda} k_3, \lambda k_4)$ actually do not depend on actual value of λ, one can also prove that $\hat{C}[\sigma\lambda g_{\vec{k}}] = \sigma\lambda\hat{C}[g_{\vec{k}}]$.

Now, the driving term of the form $D_{\sigma\lambda\vec{k}} = \sigma\lambda D_{\vec{k}}$ (spin transport induction) leads to the Boltzmann equation of the form $\hat{C}[\lambda g_{\vec{k}}] = D_{\lambda\vec{k}}$, which is the same for electrical transport induction. In other words, we state that if $\sigma\lambda g_{\vec{k}}$ is the solution for the problem of spin transport, then $\lambda g_{\vec{k}}$ is the solution for electrical transport with the same strength of force per particles. This directly leads to equality between spin conductivity and electrical conductivity at the Dirac point (in dimensionless unit) if we renormalize the electrical current and spin current to current of particles, $\sigma_S = \sigma_e$.

2.3 The collinear limit

It has been pointed out that in the case of two dimensional and linearized dispersion relation the scattering cross-section of collisions between particles diverges logarithmically for nearly parallel in-coming and out-going scatterers.[11]

The essential feature of the scattering processes at collinearity is that: for linear dispersion relation, since particles move with the same speed, if they are on the same line the interaction time will be infinite.

As discussed in Appendix B, the divergence is logarithmic. In [10], the authors showed that among other possible effects, the divergence is cut off by the fact that screening at collinear limit also becomes very strong, which actually regularizes the integral. With the purpose of just regularizing the integral we will simply include the screening of the form

$$V(q) = \frac{1}{\epsilon_s(\omega, q)} \frac{2\pi\alpha}{q}, \tag{75}$$

where

$$\epsilon_s(\omega, q) \approx 1 + \frac{\eta}{\sqrt{1 - (\omega/q)^2}}. \tag{76}$$

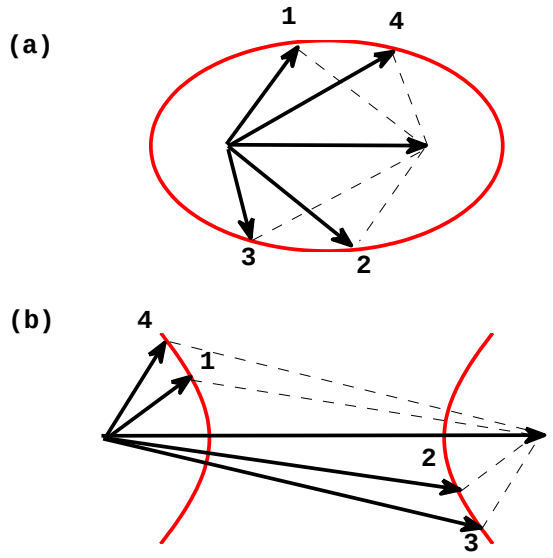


Figure 2: Diagram of scattering processes defined by energy conservation.

In the above equations, q is the modulus of momentum transferred, ω is the modulus of energy transferred and η is a small number $\eta/\alpha \approx 0.1$.

In spite of screening, the divergence is still logarithmic large when α is small. This is an interesting property of graphene, which gives a parametric justification for approximating the solution of kinetic equation. In [9] and the series [10, 11, 12], the authors pointed out and exploited the fact that in the limit, solution for the Boltzmann equation can be well estimated by a subspace of g. Indeed, in the case one can easily find out a *nearly* degenerate subspace of $\hat{C}$, which is *really* degenerate for the collinear scattering. Inversion of a *nearly* degenerate part of an operator will give the dominating contribution to the solution. We call the functions of the subspace the *main modes*.

Let us have a look at the reduced densities in the collision between particles of the same signs. The two reduced densities are

$$\mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\lambda\vec{k}_2, \sigma\lambda\vec{k}_3, \sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\sigma\lambda\vec{k}_2} - g_{\sigma\lambda\vec{k}_3} - g_{\sigma\lambda\vec{k}_4}, \tag{77}$$

$$\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\lambda\vec{k}_2, \overline{\sigma}\lambda\vec{k}_3, \sigma\lambda\vec{k}_4) = g_{\sigma\lambda\vec{k}_1} + g_{\overline{\sigma}\lambda\vec{k}_2} - g_{\overline{\sigma}\lambda\vec{k}_3} - g_{\sigma\lambda\vec{k}_4}. \tag{78}$$

In the collinear limit, the integral over the phase space of $\vec{k}_1$, $\vec{k}_2$, $\vec{q}$ will be dominated by the value of the function at $\vec{k}_1$ and $\vec{k}_2$ on nearly the same direction (and so is $\vec{q}$), say $\vec{e} = \frac{\vec{k}_1}{k_1} = \frac{\vec{k}_2}{k_2}$. On the subset of phase space, the values of $\mathcal{G}$ is simply

$$\mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\lambda\vec{k}_2, \sigma\lambda\vec{k}_3, \sigma\lambda\vec{k}_4) = \sigma\vec{e}[g_{\lambda k_1} + g_{\lambda k_2} - g_{\lambda k_3} - g_{\lambda k_4}], \tag{79}$$

$$\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\lambda\vec{k}_2, \overline{\sigma}\lambda\vec{k}_3, \sigma\lambda\vec{k}_4) = \sigma\vec{e}[g_{\lambda k_1} - g_{\lambda k_2} + g_{\lambda k_3} - g_{\lambda k_4}]. \tag{80}$$

From (79) and (80), it is easy to see that $g^0_{\lambda k} = \lambda$ and $g^0_{\lambda k} = 1$ make the reduced density $\mathcal{G}$ vanish (on the subset that we are considering). Here we would like to remind that the anti symmetry with respect to the two signs of σ in g is always required, which guarantees that the system is not polarized.[4] In the above equations, we have put the dependence on σ of the whole expression outside the bracket.

Similar arguments can be applied for the case of collisions between particles of different signs. Note, since $\vec{k}_1$ and $\vec{k}_2$ in this case are in opposite directions in the collinear limit consideration, the three reduced densities will have the form

$$\mathcal{G}(\sigma\lambda\vec{k}_1, \sigma\overline{\lambda}\vec{k}_2, \sigma\overline{\lambda}\vec{k}_3, \sigma\lambda\vec{k}_4) = \sigma\vec{e}[g_{\lambda k_1} - g_{\overline{\lambda} k_2} + g_{\overline{\lambda} k_3} - g_{\lambda k_4}], \tag{81}$$

$$\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\overline{\lambda}\vec{k}_2, \overline{\sigma}\overline{\lambda}\vec{k}_3, \sigma\lambda\vec{k}_4) = \sigma\vec{e}[g_{\lambda k_1} + g_{\overline{\lambda} k_2} - g_{\overline{\lambda} k_3} - g_{\lambda k_4}], \tag{82}$$

$$\mathcal{G}(\sigma\lambda\vec{k}_1, \overline{\sigma}\overline{\lambda}\vec{k}_2, \sigma\overline{\lambda}\vec{k}_3, \overline{\sigma}\lambda\vec{k}_4) = \sigma\vec{e}[g_{\lambda k_1} + g_{\overline{\lambda} k_2} + g_{\overline{\lambda} k_3} + g_{\lambda k_4}]. \tag{83}$$

In this case, we can see that only $g^0_{\lambda k} = \lambda$ makes the three reduced densities vanish.

According to the above arguments, the fact that $g^0_{\lambda k} = \lambda$ makes *all* the reduced densities vanish for collinear scattering means that $g^0_{\sigma\lambda\vec{k}} = \sigma\lambda\frac{\vec{k}}{k}$ itself forms a good basic for linear expansion of the solution, which implies the solution of the form $\chi(\mu)\sigma\lambda\frac{\vec{k}}{k}$.

The argument above is valid in general, and can be applied well to the case of small μ, where both collisions between particles of the same sign and of different signs are important. But let us have a look at the case of large μ. For large and positive μ (i.e small T), the collisions between particles of different signs must be negligible. Namely, we can neglect all the terms (81), (82) and (83). We are about to think that we might have two basis vector for the degenerate space according to the main modes of (79) and (80), $g^0_{\lambda k} = \lambda$ and $g^0_{\lambda k} = 1$. But again, since μ is far above the Dirac point, the difference behavior of basis vectors at λ negative does not have any considerable contribution. The two modes should give the same effect because they are the same for positive value of λ. In this sense, the two vectors are actually the same. So, to avoid degeneracy, we modify the formula for the main mode to $g^0_{\lambda k} = (1+\lambda)/2$ (a linear combination of the two modes above) for large μ.

At the Dirac point, $\mu = 0$. The main mode is $g^0_{\lambda k} = \lambda$, which explicitly reflects the symmetry between particles and holes in the system. The driving term is

$$\begin{aligned} D_0(\mu) = \langle g^0, D\rangle &= 2F_0 \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2} \{f^0_{+k}[1 - f^0_{+k}] + f^0_{-k}[1 - f^0_{-k}]\}\big|_{\mu=0} \\ &= \frac{F_0}{\pi}\int_0^{+\infty} k\mathrm{d}k[\frac{e^k}{(e^k+1)^2} + \frac{e^{-k}}{(e^{-k}+1)^2}] \\ &= \frac{2\ln 2}{\pi} F_0. \end{aligned} \tag{84}$$

[4] see Footnote 2.

For the matrix element of collision operator, we use numerical value of $C_1 = \langle g^0, \hat{C} g^0 \rangle\Big|_{\mu=0} = 1.552\alpha^2$ (see Section 2.4). With the aid of Eq. (48), we can write down the expression for spin conductivity as

$$\begin{aligned} \sigma_S &= \frac{\chi}{\pi} \int_0^{+\infty} k \mathrm{d}k \{ f^0_{+k}[1 - f^0_{+k}] + f^0_{-k}[1 - f^0_{-k}] \} \\ &= \left(\frac{2 \ln 2}{\pi} \right)^2 \frac{1}{C_1} \left[\frac{2\pi}{h} \right], \end{aligned} \tag{85}$$

where the factor inside the bracket comes when we restore the dimension of spin conductivity.

Now if we look at system at large (and positive) μ, i.e $\mu \gg 1$, we expect familiar behaviors of Fermi liquid. The following paragraphs are devoted to analysing the behavior of solution at large μ.

In this case the main mode is chosen to be

$$g^0_{\lambda k} = \frac{\lambda + 1}{2} = \begin{cases} 1 & \text{if} \quad \lambda = +1, \\ 0 & \text{if} \quad \lambda = -1. \end{cases} \tag{86}$$

Again, the matrix element of driving term D_0 can be calculated easily,

$$\begin{aligned} D_0(\mu) &= 2F_0 \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2} f^0_{+k}[1 - f^0_{+k}] \\ &= \frac{F_0}{\pi} \int_0^{+\infty} k \mathrm{d}k \frac{e^{k-\mu}}{(e^{k-\mu} + 1)^2} \\ &= \frac{F_0}{\pi} \ln(1 + e^{\mu}) \\ &\approx \frac{F_0 \mu}{\pi}. \end{aligned} \tag{87}$$

All the difficulty now is to calculate the matrix elements of collision operator, $C_{00}(\mu) = \langle g^0, \hat{C} g^0 \rangle$. Integrating the integral exactly is clearly out of hope and numerical calculation should be used. But let us have a look at the limit of very large μ, where all processes should be restricted around the Fermi surface. Indeed, in the limit $\mu \gg 1$, the main density factor F^0 will force the four vectors $(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$ move on the Fermi surface with the width $\propto 1$ (the active region). As a result, only collisions between the same signs of energy are important. The integral can be reduced to

$$\begin{aligned} C_{00}(\mu) \approx\ & 4\pi \int \frac{\mathrm{d}\vec{k}_1}{(2\pi)^2} \frac{\mathrm{d}\vec{k}_2}{(2\pi)^2} \frac{\mathrm{d}\vec{q}}{(2\pi)^2} \delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|) F^0(+k_1, +k_2, +k_3, +k_4) \\ & \times \Bigg\{ R_1(\vec{k}_1, \vec{k}_2, \vec{q}) \frac{1}{4} \left(\frac{\vec{k}_1}{k_1} + \frac{\vec{k}_2}{k_2} - \frac{\vec{k}_3}{k_3} - \frac{\vec{k}_4}{k_4} \right)^2 + \\ & + R_2(\vec{k}_1, \vec{k}_2, \vec{q}) \left[\frac{1}{4} \left(\frac{\vec{k}_1}{k_1} + \frac{\vec{k}_2}{k_2} - \frac{\vec{k}_3}{k_3} - \frac{\vec{k}_4}{k_4} \right)^2 + \frac{1}{2} \left(\frac{\vec{k}_1}{k_1} - \frac{\vec{k}_2}{k_2} + \frac{\vec{k}_3}{k_3} - \frac{\vec{k}_4}{k_4} \right)^2 \right] \Bigg\}. \end{aligned} \tag{88}$$

It turns out that actually there are two kinds of scattering that contribute to the integral. We remind that if two particles of wave-vectors $\vec{k}_1$ and $\vec{k}_2$ on the Fermi surface scatter, the out-going wave vectors will be on an ellipse as indicated on the Fig. 3. In general the ellipse crosses the active region of the Fermi surface of the order of the thickness ($\propto 1$) (see Fig. 3 (a)). As a result, only near-by-scattering, namely the out-coming wave vectors are close to the in-coming wave vectors, is allowed. But let us see what happens when the angle $\beta = (\widehat{\vec{k}_1, \vec{k}_2})$ comes closed to π, or in other words, scattering between particles in opposite side of the Fermi surface. The ellipse now is almost a circle with the radius μ (see Fig. 3 (b)). And the circle is fairly inside the active region of Fermi surface. The phenomena is like the two particles on the opposite sides of the Fermi surface (through the center) scatter into two new states, still in opposite sides but different places around the Fermi surface. This process might contribute considerably to the integral. We will call the former *near-scattering* and the latter *far-scattering*. In the following, we estimate both the contributions.

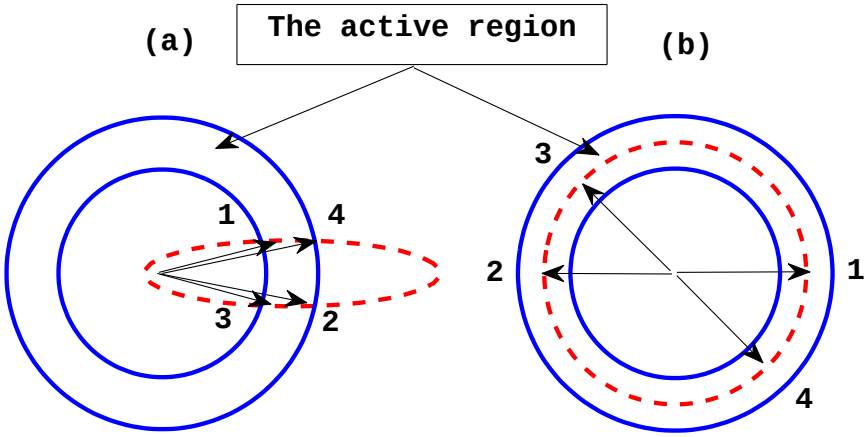


Figure 3: The diagram illustrates the scattering of electron on the Fermi surface in two case: near scattering (a) and far scattering (b).

For the *near-scattering*, at fixed $\vec{k}_1$ and $\vec{k}_2$, the phase space for $\vec{q}$ is an ellipse (see above), but only part of the ellipse is active. Crossing between the ellipse and the active region near the Fermi surface is of the order $\propto 1$. The reduced density should depend on the first order of $\propto |\vec{q}|$, to be corrected in dimension (of wave vector) we should have $\mathcal{G} \propto q/\mu$. So $\mathcal{GF} \propto q^2/\mu^2$. This scaling of $\mathcal{GF}$ will cancel the divergence in bare coulomb interaction for small q, which is $1/q^2$, and gives $\propto 1/\mu^2$. This factor again will be canceled by the phase space integration over $\vec{k}_1$ and

$\vec{k}_2$, each is scaled by the phase space of the active region over the Fermi surface $\propto \mu$. Overall we will have $C_{00}(\mu) \propto O(1)$. If we now include the Thomas-Fermi screening, the Coulomb interaction will give the scale of $\propto 1/\mu^2$. And all dependence on q will still be scaled by 1. Overall we have $C_{00}(\mu) \propto 1/\mu^2$.

Now we turn to the *far-scattering*. If we fix $\vec{k}_1$, then $\vec{k}_2$ should be considered around the opposite side of the Fermi surface. The focus of the ellipse defined above should be of the order 1 in order for it to stay inside the active region. As a result, the vector $\vec{k}_2$ is allowed to fluctuate with the order 1 on the plane. Now the phase space for $\vec{q}$ is the whole circle. On the circle, $\mathcal{G}$ will vary of the order of its values ($\propto 1$), and so does $\mathcal{GF}$. Coulomb interaction still gives the factor $\propto 1/\mu^2$. Outer integration over $\vec{k}_1$ gives phase space of order $\propto \mu$. Over all we will have the contribution of the order $\propto 1$. In other words, at large chemical potential, $C_{00}(\mu)$ simply tends to a number.

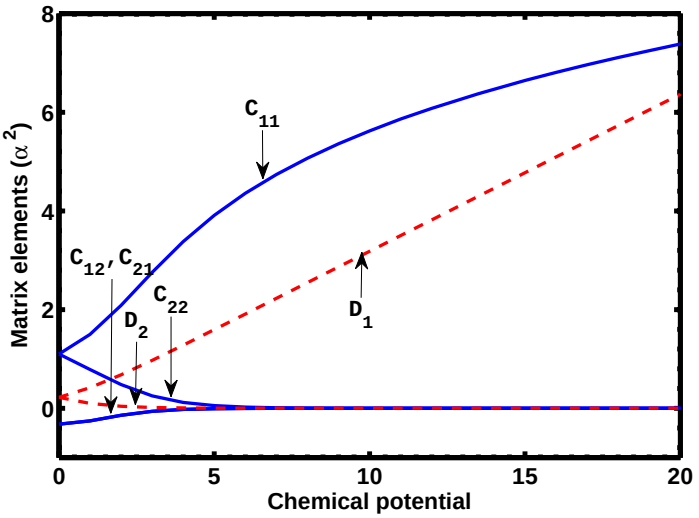


Figure 4: Matrix elements as functions of chemical potential (dimensionless units).

We see that the main contribution comes from *far-scattering* process. Actually, we can go a little bit further in estimating the integral, which gives the numerical value of $C_2 = C_{00}(+\infty)$. This estimation is given in Appendix C. Note that the approximation works only for Thomas-Fermi screening.

In this case the spin conductivity is

$$\begin{aligned} \sigma_S &= \frac{\chi}{\pi}\int_0^{+\infty} k\mathrm{d}k\{f^0_{+k}[1-f^0_{+k}]+f^0_{-k}[1-f^0_{-k}]\} \\ &= \left(\frac{\mu}{\pi T}\right)^2 \frac{1}{C_2}\left[\frac{2\pi}{h}\right]. \end{aligned} \tag{89}$$

where, again, the factor inside the bracket comes when we restore the dimension of spin conductivity. In restoring the dimension, we also change $\mu \to \mu/T$.

2.4 Numerical implementation

For the purpose of illustration, we calculated the matrix elements of $\hat{C}$ and D in the basic of two function $g^1_{\sigma\lambda\vec{k}} = \sigma\frac{\lambda+1}{2}\frac{\vec{k}}{k}$ and $g^2_{\sigma\lambda\vec{k}} = \sigma\frac{\lambda-1}{2}\frac{\vec{k}}{k}$. Variation of the matrix elements as the chemical potential varies is shown in Fig. 4.

At the Dirac point, $C_{11} = C_{22}$, this implies that the two modes are equally important. Actually the main mode is the summation of the g^1 and g^2. The matrix element we expected above $C_{00}(0)$ at $\mu = 0$ is $C_1 = \langle g^1 + g^2|\hat{C}|g^1 + g^2\rangle = C_{11} + C_{22} + C_{12} + C_{21} = 1.552\alpha^2$.

On the other hand, at large μ, all matrix elements go to zero, except for C_{11} and D_1, this illustrates the role of the main mode g^1. The matrix element of driving term (which can be calculated analytically above) linearly increases. On the other hand, the appearance of C_{11} at large μ tends to a constant, which is nothing but the value C_2 mentioned above, approximately $9.0 \div 10.0(\alpha^2)$.

The spin conductivity is calculated and plotted on Fig. 5. At the Dirac point the conductivity admits the value $0.126\alpha^{-2}$.[5] Rescaling the Fig. 5 in logarithmic scale, we can see the squared law of spin conductivity. Which actually up to $\mu = 30.0 \div 40.0$ just shows the power of ≈ 1.6. This slow convergence can be improved when we use the Thomas-Fermi screening instead.

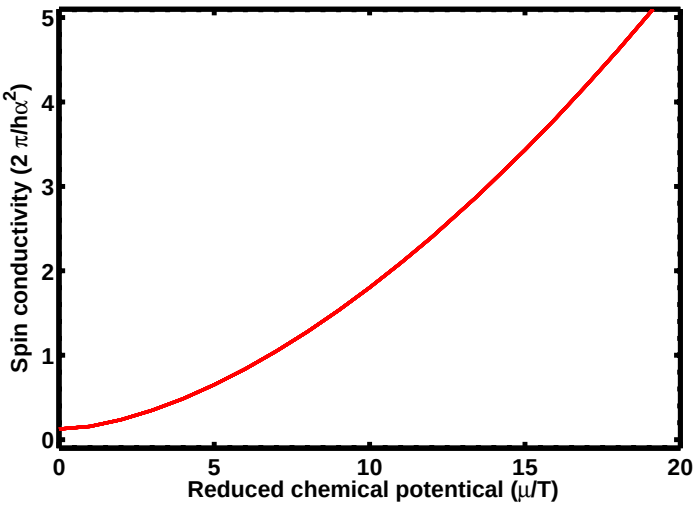


Figure 5: Spin conductivity as a function of chemical potential (full dimension units) in two basis vectors approximation.

Finally, to have a comparison, we extend the calculation to another basis. Note that the first requirement for the basis, as usual, being anti-symmetric with respect to the two signs of σ. More over, at $\mu = 0$, the basis should express the symmetry between two signs of energy too. And on the other hand, note that at large μ, only the appearance of the functions at the Fermi surface is

[5] This is different from [11] by 4 % because of the value of $\eta = 0.01$. If one restores $\eta = 0.00$, one can find the same value as there ($0.1212\alpha^{-2}$).

important. In account for such requirements, we choose the basis vectors as

$$g^n_{\sigma\lambda\vec{k}} = \sigma\lambda(k-\mu)^n\frac{\vec{k}}{k}\exp\{-\frac{(\lambda k-\mu)^2}{2\triangle^2}\}, \tag{90}$$

where $n = 0, 1, 2, ..., 6$ and $\triangle$ is a parameter for regularizing numerical integrations. Note that for $n = 0$, a part from the Gaussian factor, we find the zero modes for both small and large μ.

Cut off for integration over k is chosen to be $10.0 \div 14.0$, and so $\triangle$ is chosen to be $5.0 \div 10.0$. The spin current calculated shows a very good convergence. This gives very close value to the spin conductivity in the two-functions basis described above.

Before concluding this subsection, we note that in experiments, one usually deals with diffusion coefficient rather than spin conductivity. Taking this into account, we use the Einstein's relation to connect conductivity with diffusion coefficient. Detail analysis in Appendix D leads to the relation,

$$D_S = \sigma_S\frac{\pi}{4}\frac{(\hbar v_F)^2}{T}\frac{1}{\ln[2\mathrm{ch}\frac{\mu}{2T}]}, \tag{91}$$

where we restored all the dimensions.

Using the above analysis for the spin conductivity we see that at the Dirac point, $D_S \propto 1/T$. On the other hand, at large chemical potential, since $\sigma_S \propto \left(\frac{\mu}{T}\right)^2$ and $\ln[2\mathrm{ch}\frac{\mu}{2T}] \propto \frac{\mu}{T}$, we have $D \propto \mu/T^2$.

3 Conclusion

In summary, we have calculated the spin conductivity via the Boltzmann kinetic equation. At the Dirac point, the conductivity admits a purely interaction-limited, disorder-independent value, which also holds true for electrical transport. Away from the Dirac point, at high chemical potential, we expect the quadratic dependence of spin conductivity on the ratio μ/T. This dependence is more suitable when we have Thomas-Fermi screening, which presents for large chemical potential limit.

We note that the conductivity is calculated in the collision dominated regime. To observe the phenomena, one should exclude the effect of phonons by lowering the temperature to sufficient low value. But at low temperature, the impurity scattering in the systems is important and should be taken with care. One of the important source of impurities are the substrates on which the graphene sheet grows. The recent deposited ultrahigh mobility, suspended, single-layer graphene, might provide a chance to test the results in collision dominated regime. Note that, even we do not consider effect of phonon scattering, slow dissipation of Joule heat is necessary to avoid heating up the systems. This process is supposed to be slow, and does not contribute much to the conductivity.

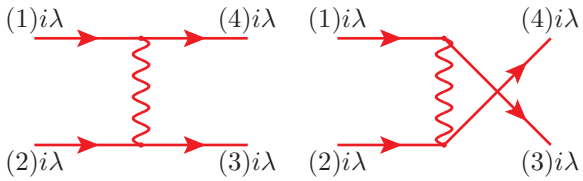
Appendices

A The matrix elements of scattering processes

First of all, the Hamiltonian H_1 conserves the spins of particles in collisions. Moreover, to satisfy the energy conservation and momentum conservation simultaneously one can infer another "law of conservation", namely the conservation of net sign of energy. Indeed, if the sign of energy is not conserved, one can not have the momentum conservation and energy conservation simultaneously. Note that despite the net sign of in-coming particles equals to net sign of out-going particles, the signs of them can be flipped in scattering processes (of course, this is meaningful only when we can distinguish them by "flavors").

Then the matrix elements of scattering process can be divided in five classes due to "flavors" and signs of the fermions in the process.

1. Scattering of particles of the same signs of energy and the same flavors:



$$|i\lambda\vec{k}_1, i\lambda\vec{k}_2\rangle \rightarrow |i\lambda\vec{k}_4, i\lambda\vec{k}_3\rangle \tag{92}$$

Operator presentation for the states will be

$$\begin{aligned} |A\rangle = |i\lambda\vec{k}_1, i\lambda\vec{k}_2\rangle &= \gamma^\dagger_{i\lambda\vec{k}_2}\gamma^\dagger_{i\lambda\vec{k}_1}|0\rangle, \\ |B\rangle = |i\lambda\vec{k}_4, i\lambda\vec{k}_3\rangle &= \gamma^\dagger_{i\lambda\vec{k}_3}\gamma^\dagger_{i\lambda\vec{k}_4}|0\rangle. \end{aligned} \tag{93}$$

The matrix element is

$$\langle B|H_1|A\rangle = \sum T_{\lambda'_1\lambda'_2\lambda'_3\lambda'_4}(\vec{k}'_1, \vec{k}'_2, \vec{k}'_3, \vec{k}'_4)\langle 0|\gamma_{i\lambda\vec{k}_4}\gamma_{i\lambda\vec{k}_3}\gamma^\dagger_{i'\lambda'_4\vec{k}'_4}\gamma^\dagger_{j'\lambda'_3\vec{k}'_3}\gamma_{j'\lambda'_2\vec{k}'_2}\gamma_{i'\lambda'_1\vec{k}'_1}\gamma^\dagger_{i\lambda\vec{k}_2}\gamma^\dagger_{i\lambda\vec{k}_1}|0\rangle, \tag{94}$$

where we remind that $\vec{k}'_3 = \vec{k}'_2 - \vec{q}'$ and $\vec{k}'_4 = \vec{k}'_1 + \vec{q}'$ and the summation is taken over all i', j', λ'_1, λ'_2, λ'_3, λ'_4, $\vec{k}'_1$, $\vec{k}'_2$, $\vec{q}'$.

Now we look for the non-zero terms in the summation. In order to be non-zero, the terms should have $i' = j' = i$, $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$, and $(\{\vec{k}'_1, \vec{k}'_2\} = \{\vec{k}_1, \vec{k}_2\}, \{\vec{k}'_3, \vec{k}'_4\} = \{\vec{k}_3, \vec{k}_4\})$. Permutation of momenta gives four terms, and note that fermionic creators and annihilators are anti-symmetric under permutations, then

$$\begin{aligned} \langle B|H_1|A\rangle &= T_{\lambda\lambda\lambda\lambda}(1,2,3,4) + T_{\lambda\lambda\lambda\lambda}(2,1,4,3) - T_{\lambda\lambda\lambda\lambda}(1,2,4,3) - T_{\lambda\lambda\lambda\lambda}(2,1,3,4) \\ &= 2[T_{++++}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) - T_{++++}(\vec{k}_1, \vec{k}_2, \vec{k}_4, \vec{k}_3)], \end{aligned} \tag{95}$$

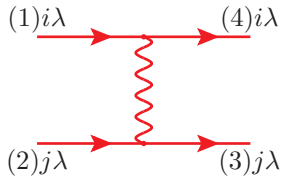
where we dropped the explicit vector symbols in the intermediate steps. The fact that $T_{\lambda_1\lambda_2\lambda_3\lambda_4}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ remains the same if we change the signs of particles $\{+\to-;-\to+\}$ has been used. Moreover the function is also symmetric under exchanging $1\to2, 2\to1, 3\to4, 4\to3$. Both of the properties will also be used frequently in the latter cases.

In this case, spins of the particle are the same, when integrate over the all phase space we should have double-counted the phase space. Instead, we will divide the rate by 2 and have the final formula,

$$R_1(\vec{k}_1,\vec{k}_2,\vec{q}) = 2\left|T_{++++}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) - T_{++++}(\vec{k}_1,\vec{k}_2,\vec{k}_4,\vec{k}_3)\right|^2. \tag{96}$$

Similar arguments will be applied for the other four cases as follows.

2. Scattering of particles of the same sign of energy but different flavors:

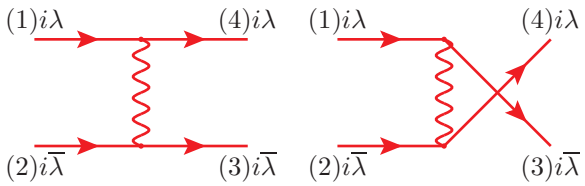


$$|i\lambda\vec{k}_1, j\lambda\vec{k}_2\rangle \to |i\lambda\vec{k}_4, j\lambda\vec{k}_3\rangle \tag{97}$$

The non-zero terms should have $(\{i'\lambda_1'\vec{k}_1'\} = \{i\lambda\vec{k}_1\}, \{j'\lambda_2'\vec{k}_2'\} = \{j\lambda\vec{k}_2\}, \{j'\lambda_3'\vec{k}_3'\} = \{j\lambda\vec{k}_3\}, \{i'\lambda_4'\vec{k}_4'\} = \{i\lambda\vec{k}_4\})$ or $(\{i'\lambda_1'\vec{k}_1'\} = \{j\lambda\vec{k}_2\}, \{j'\lambda_2'\vec{k}_2'\} = \{i\lambda\vec{k}_1\}, \{j'\lambda_3'\vec{k}_3'\} = \{i\lambda\vec{k}_4\}, \{i'\lambda_4'\vec{k}_4'\} = \{j\lambda\vec{k}_3\})$, both give the same amplitude. The rate would be

$$R_2(\vec{k}_1,\vec{k}_2,\vec{q}) = 4\left|T_{++++}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\right|^2. \tag{98}$$

3. Scattering of particles of the different signs of energy and the same flavor:

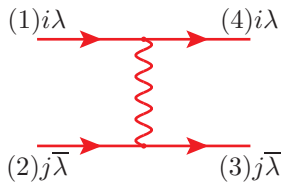


$$|i\lambda\vec{k}_1, i\overline{\lambda}\vec{k}_2\rangle \to |i\lambda\vec{k}_4, i\overline{\lambda}\vec{k}_3\rangle \tag{99}$$

The non-zero terms should have $i' = j' = i$, $(\{\lambda_1'\vec{k}_1', \lambda_2'\vec{k}_2'\} = \{\lambda\vec{k}_1, \overline{\lambda}\vec{k}_2\})$ and $(\{\lambda_3'\vec{k}_3', \lambda_4'\vec{k}_4'\} = \{\overline{\lambda}\vec{k}_3, \lambda\vec{k}_4\})$. Taking account for changing of sign in permutation, we come to the rate

$$R_3(\vec{k}_1,\vec{k}_2,\vec{q}) = 4\left|T_{+--+}(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) - T_{+-+-}(\vec{k}_1,\vec{k}_2,\vec{k}_4,\vec{k}_3)\right|^2. \tag{100}$$

4. Scattering of particles of different signs of energy and the different flavors:

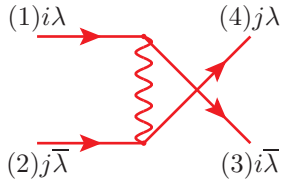


$$|i\lambda\vec{k}_1, j\overline{\lambda}\vec{k}_2\rangle \to |i\lambda\vec{k}_4, j\overline{\lambda}\vec{k}_3\rangle \tag{101}$$

The non zero-terms should have ($\{i'\lambda_1'\vec{k}_1'\} = \{i\lambda\vec{k}_1\}$, $\{j'\lambda_2'\vec{k}_2'\} = \{j\overline{\lambda}\vec{k}_2\}$, $\{j'\lambda_3'\vec{k}_3'\} = \{j\overline{\lambda}\vec{k}_3\}$, $\{i'\lambda_4'\vec{k}_4'\} = \{i\lambda\vec{k}_4\}$) or ($\{i'\lambda_1'\vec{k}_1'\} = \{j\overline{\lambda}\vec{k}_2\}$, $\{j'\lambda_2'\vec{k}_2'\} = \{i\lambda\vec{k}_1\}$, $\{j'\lambda_3'\vec{k}_3'\} = \{i\lambda\vec{k}_4\}$, $\{i'\lambda_4'\vec{k}_4'\} = \{j\overline{\lambda}\vec{k}_3\}$), both give the same amplitude. The rate would be

$$R_4(\vec{k}_1, \vec{k}_2, \vec{q}) = 4\left|T_{+--+}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)\right|^2. \tag{102}$$

5. Scattering of particles of different signs of energy and different flavor, signs exchanging:



$$|i\lambda\vec{k}_1, j\overline{\lambda}\vec{k}_2\rangle \to |j\lambda\vec{k}_4, i\overline{\lambda}\vec{k}_3\rangle \tag{103}$$

The non-zero term should have ($\{i'\lambda_1'\vec{k}_1'\} = \{i\lambda\vec{k}_1\}$, $\{j'\lambda_2'\vec{k}_2'\} = \{j\overline{\lambda}\vec{k}_2\}$, $\{j'\lambda_3'\vec{k}_3'\} = \{i\overline{\lambda}\vec{k}_3\}$, $\{i'\lambda_4'\vec{k}_4'\} = \{j\lambda\vec{k}_4\}$) or ($\{i'\lambda_1'\vec{k}_1'\} = \{j\overline{\lambda}\vec{k}_2\}$, $\{j'\lambda_2'\vec{k}_2'\} = \{i\lambda\vec{k}_1\}$, $\{j'\lambda_3'\vec{k}_3'\} = \{j\lambda\vec{k}_4\}$, $\{i'\lambda_4'\vec{k}_4'\} = \{i\overline{\lambda}\vec{k}_3\}$), both give the same amplitude. The rate would be

$$R_5(\vec{k}_1, \vec{k}_2, \vec{q}) = 4\left|T_{+-+-}(\vec{k}_1, \vec{k}_2, \vec{k}_4, \vec{k}_3)\right|^2. \tag{104}$$

B Parametrizing the energy conservation

Let us begin with scattering between particles of the same signs. As noted in section 2.3, in the case of collisions between particles of the same signs, the out-going momenta are on an ellipse defined by the in-coming particles. This inspires us to use the elliptic coordinate to parametrize the energy conserving function.[12]

We have $|\vec{k}_3 + \vec{k}_4| = 2f = |\vec{k}_1 + \vec{k}_2|$ and $k_3 + k_4 = 2a = k_1 + k_2$, where f is the *focus* of the ellipse, and a is the *semi-major-radius*.

Noting that $\vec{q} = \vec{k}_4 - \vec{k}_1 = \vec{k}_2 - \vec{k}_3$, let us define $\vec{p} = \vec{k}_4 - \frac{1}{2}(\vec{k}_1 + \vec{k}_2)$ then $\vec{q} = \vec{p} - \frac{1}{2}(\vec{k}_1 - \vec{k}_2)$. Now we use the elliptic coordinates to indicate $\vec{p}$, $\vec{p} = p_1\vec{e}_1 + p_2\vec{e}_2 = a'\cos\phi\vec{e}_1 + b'\sin\phi\vec{e}_2$, where $\vec{e}_1 = (\vec{k}_1 + \vec{k}_2)/|\vec{k}_1 + \vec{k}_2|$, $\vec{e}_2 = [\vec{e}_z \times \vec{e}_1]$ and the relation between a' and b' is $b' = \sqrt{a'^2 - f^2}$. In these parameters, we have $k_3 + k_4 = 2a'$. According to these transformations, an integral of the form

$$I_1 = \int \mathrm{d}\vec{q}\delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|)\mathcal{F}(\vec{k}_1, \vec{k}_2, \vec{q}) \tag{105}$$

would become

$$I_1 = \int_0^{+\infty} \mathrm{d}a' \int_0^{2\pi} \mathrm{d}\phi \frac{\partial(p_1,p_2)}{\partial(a',\phi)} \delta(2a'-2a) \mathcal{F}(\vec{k}_1, \vec{k}_2, a'\cos\phi\vec{e}_1 + b'\sin\phi\vec{e}_2 - \frac{1}{2}(\vec{k}_1 - \vec{k}_2)), \quad (106)$$

where the Jacobian factor is

$$\frac{\partial(p_1,p_2)}{\partial(a',\phi)} = \begin{vmatrix} \cos\phi & -a'\sin\phi \\ \frac{a'}{b'}\sin\phi & b'\cos\phi \end{vmatrix} = \frac{a'^2 - f^2\cos^2\phi}{b'}. \quad (107)$$

The effect of δ - function now is simply to replace $a' \to a$ and give a factor $1/2$ outside. And so

$$I_1 = \frac{1}{2}\int_0^{2\pi} \mathrm{d}\phi \frac{a^2 - f^2\cos^2\phi}{b} \mathcal{F}(\vec{k}_1, \vec{k}_2, a\cos\phi\vec{e}_1 + b\sin\phi\vec{e}_2 - \frac{1}{2}(\vec{k}_1 - \vec{k}_2)). \quad (108)$$

Actually, if we perform the integral over $\vec{k}_1$ and $\vec{k}_2$ as well, the integral would look like $J_1 = \int \mathrm{d}\vec{k}_1 \mathrm{d}\vec{k}_2 I_1(\vec{k}_1, \vec{k}_2)$. The function $I_1(\vec{k}_1, \vec{k}_2)$ always depends on the relative direction between them only, so it is convenient to choose the basic vectors as: $\vec{e} = \vec{k}_1/k_1$ and $\vec{n} = [\vec{e}_z \times \vec{e}]$. Expansions of the vectors $\vec{k}_1$, $\vec{k}_2$ on the basic will be $\vec{k}_1 = k_1\vec{e}$, $\vec{k}_2 = k_2\cos\beta\vec{e} + k_2\sin\beta\vec{n}$, where β is the angle between $\vec{k}_2$ and $\vec{k}_1$. Then performing the expansion of $\vec{q}$ in the basis vector too, we finally have the integral

$$J_1 = 2\pi \int_0^{+\infty} k_1 \mathrm{d}k_1 \int_0^{+\infty} k_2 \mathrm{d}k_2 \int_0^{2\pi} \mathrm{d}\beta \int_0^{2\pi} \mathrm{d}\phi \frac{a^2 - f^2\cos^2\phi}{2b} \mathcal{F}(k_1\vec{e}, k_2\cos\beta\vec{e} + k_2\sin\beta\vec{n}, A\vec{e} + B\vec{n}), \quad (109)$$

where:

$$\begin{aligned} a &= \frac{1}{2}(k_1 + k_2), \\ f &= \frac{1}{2}\sqrt{k_1^2 + k_2^2 + 2k_1k_2\cos\beta}, \\ b &= \frac{1}{2}k_1k_2\sin\beta/2, \\ A &= (k_1 + k_2\cos\beta)\frac{a}{2f}\cos\phi - k_2\sin\beta\frac{b}{2f}\sin\phi + \frac{1}{2}(k_2\cos\beta - k_1), \\ B &= k_2\frac{a}{2f}\sin\beta\cos\phi + (k_1 + k_2\cos\beta)\frac{b}{2f}\sin\phi + \frac{1}{2}k_2\sin\beta. \end{aligned} \quad (110)$$

In paramerizing the integral we can see clearly the logarithmic divergence of the integral according to β. Indeed, near $\beta = 0$ we have $b \propto \sin\beta/2 \approx \beta/2$ which makes the integral divergent. As mentioned above, this divergence can be removed by including screening effect and on the other hand gives a parametrically justified approximation dominated by the collinear limit.

Turning to scattering of particles of different signs, one can also do the similar parametrization for hyperbola curve of energy conservation. But the more simple way to do is to map them into collisions between particles of the same sign as what we have done in Section 2.2. Then all one have to do is to use the parametrization above.

C The matrix element of collision operator: large chemical potential limit

We are to estimate the matrix element of collision operator in the collinear limit at large chemical potential limit, i.e Eq. (88). For large chemical potential, screening is preferred to be of Thomas-Fermi form,

$$V(q) = \frac{2\pi\alpha}{q+q_0}, \tag{111}$$

where $q_0 = \eta_0\mu$ is the Thomas-Fermi wave vector, and η_0 is of order of α.

The first note is that the term like $\left(\frac{\vec{k}_1}{k_1} + \frac{\vec{k}_2}{k_2} - \frac{\vec{k}_3}{k_3} - \frac{\vec{k}_4}{k_4}\right)^2$ resembles the momentum conservation and should be small when all the vectors are almost of the same length. The main contribution should come from the term containing $\left(\frac{\vec{k}_1}{k_1} - \frac{\vec{k}_2}{k_2} + \frac{\vec{k}_3}{k_3} - \frac{\vec{k}_4}{k_4}\right)^2$,

$$\begin{aligned} C_2 = C_{00}(\mu) \quad \approx \quad & 2\pi \int \frac{\mathrm{d}\vec{k}_1}{(2\pi)^2}\frac{\mathrm{d}\vec{k}_2}{(2\pi)^2}\frac{\mathrm{d}\vec{q}}{(2\pi)^2}\delta(|\vec{k}_1| + |\vec{k}_2| - |\vec{k}_3| - |\vec{k}_4|)F^0(+k_1, +k_2, +k_3, +k_4) \\ & \times R_2(\vec{k}_1, \vec{k}_2, \vec{q})\left(\frac{\vec{k}_1}{k_1} - \frac{\vec{k}_2}{k_2} + \frac{\vec{k}_3}{k_3} - \frac{\vec{k}_4}{k_4}\right)^2. \end{aligned} \tag{112}$$

Now, we make the statement that the main factor of density F^0 force all the vectors $\vec{k}_1$, $\vec{k}_2$, $\vec{k}_3$ and $\vec{k}_4$ move on the Fermi surface more precise. That means we expect the equality

$$\delta(k_1 + k_2 - k_3 - k_4)F^0(+k_1, +k_2, +k_3, +k_4) = C\delta(k_1 - \mu)\delta(k_2 - \mu)\delta(k_3 - \mu)\delta(k_4 - \mu), \tag{113}$$

where C is a coefficient we will calculate right now. The way to fix C is to do the integral on the both sides of the equation over all the variables k_1, k_2, k_3 and k_4.After changing the variables $p_1 = k_1 - k_4$, $p_2 = k_2 - k_3$, the effect of delta function now is simply to replace p_1 by $-p$ and p_2 by $+p$ and the integral becomes

$$C = \int_0^{+\infty}\mathrm{d}k_1\int_0^{+\infty}\mathrm{d}k_2\int_{-\infty}^{+\infty}\mathrm{d}p\frac{1}{(e^{k_1-\mu}+1)(e^{k_2-\mu}+1)(e^{-k_1-p+\mu}+1)(e^{-k_2+p+\mu}+1)}. \tag{114}$$

Now, note that

$$\begin{aligned} \int_0^{+\infty}\mathrm{d}k_1\frac{1}{e^{k_1-\mu}+1}\frac{1}{e^{-k_1-p+\mu}+1} &= \frac{1}{1-e^{-p}}\ln\frac{1+e^{\mu}}{1+e^{\mu-p}} \approx \frac{p}{1-e^{-p}}, \\ \int_0^{+\infty}\mathrm{d}k_2\frac{1}{e^{k_2-\mu}+1}\frac{1}{e^{-k_2+p+\mu}+1} &= \frac{1}{1-e^{+p}}\ln\frac{1+e^{\mu}}{1+e^{\mu+p}} \approx \frac{p}{e^{+p}-1}, \end{aligned}$$

and then

$$C = \int_{-\infty}^{+\infty}\mathrm{d}p\frac{p^2}{(e^{+p}-1)(1-e^{-p})} = \frac{2\pi^2}{3}. \tag{115}$$

The δ-functions in the right-hand side of Eq. (113) mean that in all the expression afterwards we can replace k_1, k_2, k_3, k_4 by μ. Now we call β the angle between $\vec{k}_2$ and $\vec{k}_1$, ϕ the angle between $\vec{k}_4$ and $\vec{k}_1$. with the aid of β and ϕ we write, we are going to parametrize the integral.

Momentum conservation gives $\vec{k}_3 = \vec{k}_1 + \vec{k}_2 - \vec{k}_4$. Moreover, because μ and k_3 are both positive, we can have $\delta(|\vec{k}_3| - \mu) = 2\mu\delta(\vec{k}_3^2 - \mu)$. And so

$$\begin{aligned}
\delta(|\vec{k}_3| - \mu) &= 2\mu\delta(\vec{k}_3^2 - \mu^2) \\
&= 2\mu\delta([\vec{k}_1 + \vec{k}_2 - \vec{k}_4]^2 - \mu^2) \\
&= 2\mu\delta(\vec{k}_1^2 + \vec{k}_2^2 + \vec{k}_4^2 + 2\vec{k}_1\vec{k}_2 - 2\vec{k}_1\vec{k}_4 - 2\vec{k}_2\vec{k}_4 - \mu^2) \\
&= \frac{1}{\mu}\delta(1 + \cos\beta - \cos\phi - \cos(\phi - \beta)) \\
&= \frac{1}{\mu}\delta(4\cos\frac{\beta}{2}\sin\frac{\phi}{2}\sin\frac{\phi - \beta}{2}).
\end{aligned} \tag{116}$$

One can check that in the case $\phi = 0$, $\phi - \beta = 0$ the vector $\vec{k}_3$ and $\vec{k}_4$ are the same as $\vec{k}_1$ and $\vec{k}_2$, so the factor $\mathcal{GF}$ identically vanishes. Then we just have to consider the line $\beta = \pi$. Let us call $f(\beta, \phi) = 1 + \cos\beta - \cos\phi - \cos(\phi - \beta)$, the gradient of $g(\beta, \phi)$ is

$$\left\{\begin{array}{lcl} \frac{\partial f}{\partial \beta} &=& -\sin\beta - \sin(\phi - \beta), \\ \frac{\partial f}{\partial \phi} &=& +\sin\phi + \sin(\phi - \beta). \end{array}\right. \tag{117}$$

In effect, we can write [17]

$$\delta(|\vec{k}_3| - \mu) = \frac{1}{|\nabla f(\beta, \phi)|}\delta(\beta - \pi) = \frac{1}{\mu}\frac{1}{|\sin\phi|}\delta(\beta - \pi). \tag{118}$$

Note that this formula is verified only when the gradient of the function $f(\beta, \phi)$ has no singular point. In our case, $\phi = 0$ is a singular point of the distribution. Fortunately the function $\mathcal{GF}$ regularizes this singular (as discussed above). In other word, one can cut off the integral near the singular point, then let the cutoff tends to zero yielding a finite limit for the integral.

For the rate, we begin with the estimation for $T_{++++}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4)$ as follows

$$\begin{aligned}
T_{++++}(\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4) &= \frac{V(q)}{8}\left[1 + \lambda_4\lambda_1\frac{K_4^*}{|K_4|}\frac{K_1}{|K_1|}\right]\left[1 + \lambda_3\lambda_2\frac{K_3^*}{|K_3|}\frac{K_2}{|K_2|}\right] \\
&\approx \frac{V(q)}{8}(1 + e^{-i\phi})(1 + e^{-i\phi}),
\end{aligned} \tag{119}$$

where Coulomb interaction with Thomas-Fermi screening is of the form

$$V(q) = \frac{2\pi\alpha}{q + q_0} = \frac{2\pi\alpha}{\mu[2\sin\phi/2 + \eta_0]}. \tag{120}$$

Then, from Eqs. (119) and (120) we have

$$R_2(\vec{k}_1, \vec{k}_2, \vec{q}) = \frac{1}{\mu^2}\frac{(2\pi\alpha)^2}{(2\sin\phi/2 + \eta_0)^2}\cos^4\phi/2. \tag{121}$$

Finally, we can estimate the reduced densities as

$$\left(\frac{\vec{k}_1}{k_1} - \frac{\vec{k}_2}{k_2} + \frac{\vec{k}_3}{k_3} - \frac{\vec{k}_4}{k_4}\right)^2 \approx \left(\frac{-2\vec{q}}{\mu}\right)^2 = 16\sin^2\phi/2. \tag{122}$$

And over all, inserting Eqs. (115), (121), (122) into (112) we have

$$C_2 = \frac{8\alpha^2}{3}\int_0^{2\pi} \mathrm{d}\phi \frac{1}{|\sin\phi|}\frac{\cos^4\phi/2}{(2\sin\phi/2+\eta_0)}\sin^2\phi/2. \tag{123}$$

After some transformations we would have

$$C_2 = \frac{16\alpha^2}{3}\int_0^1 \mathrm{d}x \frac{x(1-x^2)}{[2x+\eta_0]^2}. \tag{124}$$

The integral is elementary and can be calculated easily. By inserting the δ-function into the place of Fermi distributions, one has to verify that the function under the integral is smooth enough in compare to the broaden δ-function. This approximation is verified for Thomas-Fermi screening, where one does not encounter with singular point of Coulomb potential. On the other hand, for bare Coulomb interaction, it becomes a divergent integral, which reflects the bad behavior of function over the active region of the Fermi surface.

D The diffusion coefficient: Einstein's relation

First of all we begin with calculation for the paramagnetic susceptibility of Dirac gas on the plane, κ. In this section, we will begin with SI unit system instead of dimensionless unit.

Before begin rigorous calculation, let us check out the dependence of κ on T. The spin polarization is proportional to the number of active electron about the Dirac cone at temperature T, which in turn is scaled as $\propto T^2$. On the other hand, the probability difference between up and down states of spin is proportional to B/T. So overall $\kappa \propto T$.

For each electron at state $(\lambda, \vec{k})$, there are two possibilities for spin, i.e up and down with different energy by $E_\sigma = -\mu_B B\sigma$. This result in polarization of spin,

$$P_S = 2\sum_\lambda \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2}\left\{\frac{1}{e^{(\lambda\hbar v_F k-\mu-\mu_B B)/T}+1} - \frac{1}{e^{(\lambda\hbar v_F k-\mu+\mu_B B)/T}+1}\right\}, \tag{125}$$

where the factor 2 comes from the valley degeneracy. Expanding and keeping first order of B in the expression under the integration, one have the simplified formula,

$$P_S = 2\sum_\lambda \int \frac{\mathrm{d}\vec{k}}{(2\pi)^2}\frac{e^{(\lambda\hbar v_F k-\mu)/T}}{[e^{(\lambda\hbar v_F k-\mu)/T}+1]^2}\frac{2\mu_B B}{T}. \tag{126}$$

To see the meaning of scaling, we change the unit for wave vector to $T/\hbar v_F$.This integral can be rewritten in dimensionless unit of integrated variable k,

$$P_S = \frac{2}{\pi}\frac{\mu_B B}{T}\left(\frac{T}{\hbar v_F}\right)^2 \sum_\lambda \int_0^\infty k\mathrm{d}k \frac{e^{\lambda k-\mu/T}}{[e^{\lambda k-\mu/T}+1]^2}. \tag{127}$$

The integral can be found without difficulties, and finally we can derive formula for the magnetic susceptibility,

$$\kappa = \frac{P_S}{B} = \frac{4}{\pi}\frac{\mu_B T}{(\hbar v_F)^2}\ln[2\mathrm{ch}\frac{\mu}{2T}]. \tag{128}$$

With the aid of the spin susceptibility calculated above, we will show that the diffusion coefficient can be obtained from the spin conductivity via the so-call Einstein relation. Suppose we have inhomogeneous magnetic field over the plane, this results in inhomogeneous spin polarization over the sheet,

$$P_S = \kappa B, \tag{129}$$

where κ is the Pauli susceptibility (of spin) of Dirac gas on the graphene sheet. Inhomogeneous polarization generates diffusion current,

$$j_{\text{diff.}} = -D_S \nabla P_S, \tag{130}$$

where D_S is the spin diffusion coefficient. In equilibrium, diffusion current is compensated by current induced by external field, $j_{\text{diff.}} + j_{\text{ext.}} = 0$. The current due to external field, according to above argument is found to be

$$j_{\text{ext.}} = \sigma_S \mu_B \nabla B, \tag{131}$$

where σ_S is the spin conductivity and μ_B is the magneton Bohr. Then we can derive the formula for diffusion coefficient,

$$D_S = \sigma_S \frac{\mu_B}{\kappa}. \tag{132}$$

Insert the formula for susceptibility obtained above in Eq. (128), we reach to

$$D_S = \sigma_S \frac{\pi}{4} \frac{(\hbar v_F)^2}{T} \frac{1}{\ln[2\text{ch}\frac{\mu}{2T}]}. \tag{133}$$

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