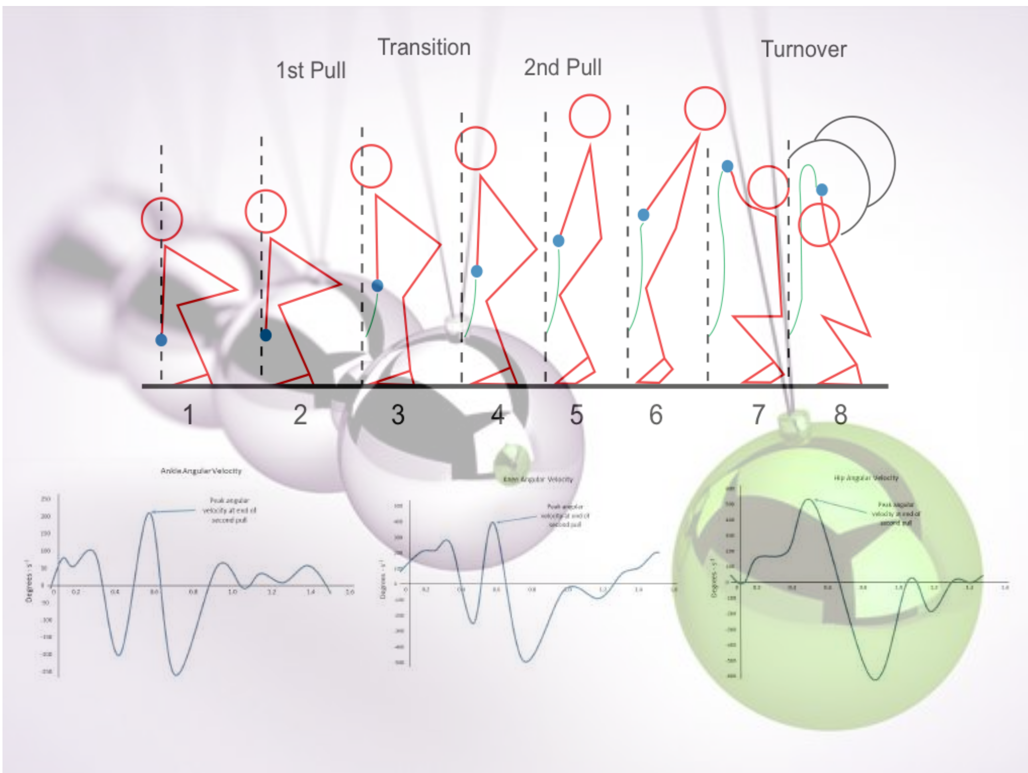
Foundations of Physics 1
Mechanics 2

Michaelmas 2018



Per Aspera ad Astra

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IPPP Durham

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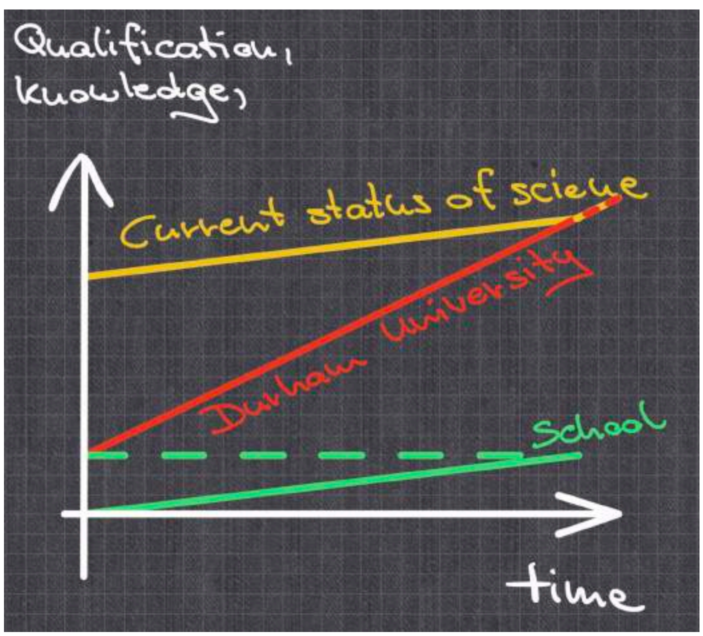
Contents

1 Introduction

Dear students,
I am very much looking forward to teach this course. I am professor for theoretical physics at Durham University and my specialisation is elementary particle physics (see https://www.ippp.dur.ac.uk/profile/lenz for more information).
The lectures will closely follow the textbook *University Physics with Modern Physics, 13th edition, Young and Freedman, customised edition for Durham University* - for each lecture the corresponding pages in the text-book are listed under the section headline and I would like to recommend to you to have a look at the relevant pages before each lecture.
I will also prepare these LATEX notes, which will be made available to you on DUO some days **after** the lecture. **Why after?** In the coming years at Durham University you will work at a pace that is probably considerably higher than what you might have been used to from school - we want to bring you to a level that is close to the current status of science. If you would only be sitting in the lectures and letting yourself be entertained by the lecturer then we will not get very far.



I consider myself more like a coach for you, I am trying to explain you the

relevant concepts in the lecture, but in order to really understanding them you will have to work by yourself[1] - e.g. solving exercises (your textbook contains a huge number of them), redoing derivations from the lectures,... . My experience is that students profit much more from a lecture, when they attend and they write the notes from the white/blackboard by themselves. So the aim of these notes is not to release you from the task of writing during the lecture, but it is to support you in the preparation for your exams by providing a copy of the notes on the white/blackboard without any typos. To achieve this, I would like to ask you to inform me about any misprints that you find in the notes on DUO. For similar reasons the lectures will also not be recorded - at least not until the recording system is working perfectly. If you want to contact me, my email address is alexander.lenz@durham.ac.uk and my office number is OC121 in the "old" Ogden Building, which is attached to the physics building.
Finally have fun with Mechanics and do not underestimate this topic - even if many things seem to be very familiar or easy. Mechanics contains an amazing amount of concepts that appear later in e.g. Quantum Mechanics, Quantum Field Theory, Elementary Particle Theory and Cosmology.
As already said I am really looking forward in guiding you through this topic.

Prof. Alexander Lenz

[1]A weightlifter will also not get stronger by only listening to some advice, he will actually have to train hard by himself. But by getting good advice he will get stronger very fast; by getting no advise or the wrong one, he will probably struggle. Some confirming evidence for that can also be found on my homepage.

2 Lecture 1: Rotation of Rigid Bodies 1

Textbook pages 278- 288, Section 9.1 - 9.3
Revise Textbook pages 85- 87, Section 3.4

2.1 Angular Velocity and Angular Acceleration

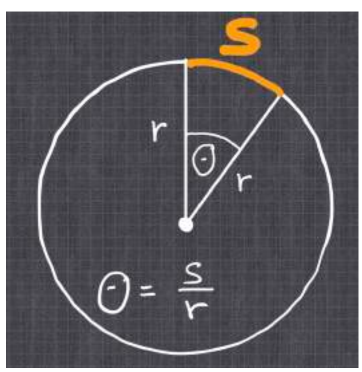
Measurements of angles are typically done in **degrees**. A full circle corresponds to an angle of 360°.
We will use new units for angles, called **radians**. In these units a full circle corresponds to an angle of 2π. Thus we get[2]

$$\frac{\theta^{in\,degrees}}{360^\circ} = \frac{\theta^{in\,radians}}{2\pi} \quad \Rightarrow \quad \begin{cases} \theta^{in\,degrees} = \frac{360^\circ}{2\pi}\theta^{in\,radians} \\ \theta^{in\,radians} = \frac{2\pi}{360^\circ}\theta^{in\,degrees} \end{cases} . \tag{1}$$

Example L1.1:

$$\theta^{in\,degrees} = 1^\circ \quad \Leftrightarrow \quad \theta^{in\,radians} = \frac{2\pi}{360^\circ} \cdot 1^\circ \approx 0.017\,.$$
$$\theta^{in\,radians} = 1 \quad \Leftrightarrow \quad \theta^{in\,degrees} = \frac{360^\circ}{2\pi} \cdot 1 \approx 57.3^\circ\,.$$

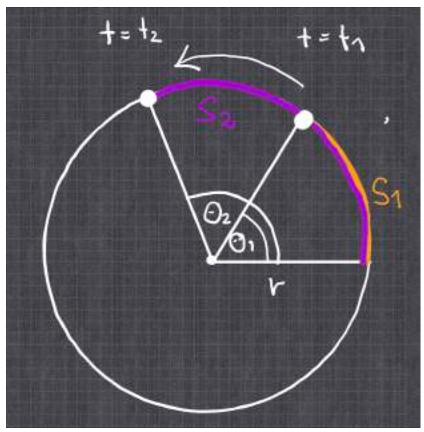
Angles measured in radians can also be visualised as the segment of the circle s that corresponds to the angle θ divided by the radius of the circle (or equivalently as the segment of the unit circle).



[2]We denote angles by the greek letter θ (theta).

If a particle is moving on a circle around the z-axis, then we define its angular velocity[3] as angle per time. If the particle sits at time t_1 at the position defined by the angle θ_1 and at time t_2 at the position defined by the angle θ_2, then the average **angular velocity** is defined as:

$$\omega_{average,z} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}\,; \qquad \Delta t \to 0 : \, \omega_z = \frac{d\theta}{dt}\,. \tag{2}$$



Example L1.2: The blade of a circular saw spins with 5000 RPM (Revolutions per Minute); determine the angular velocity!

$$\omega = \frac{5000 \cdot 2\pi}{60s} = 523.6s^{-1}\,.$$

Remember: to determine the angular velocity in the unit s^{-1}, you have to measure the angles in radians and not degrees! If you want to use degrees for the measurement of angles, then your angular velocity will have the unit $^\circ/s$.

The vector of the angular velocity $\vec{\omega}$ is directed along the rotation axis - the sign of $\vec{\omega}$ can be determined by the right hand rule.

[3] We denote the angular velocity by the greek letter ω (omega).

In analogy to the case of a linear movement we define the **angular acceleration**[4] as

$$\alpha_{average,z} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}\,; \qquad \Delta t \to 0:\ \alpha_z = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}\,. \tag{3}$$

The vector of angular acceleration $\vec{\alpha}$ is parallel to $\vec{\omega}$, if the rotation is speeding up and anti-parallel to $\vec{\omega}$ if the rotation is slowing down.

Example L1.3: An angular movement is given by

$$\begin{aligned}\theta(t) &= \pi + \omega_0 t + \frac{1}{2}\alpha t^2 \\ &= \pi + 7.5\frac{1}{s}t + 45\frac{1}{s^2}t^2\,.\end{aligned}$$

Determine the angular velocity and the angular acceleration!

$$\begin{aligned}\omega(t) &= \frac{d\theta(t)}{dt} = 7.5\frac{1}{s} + 90\frac{1}{s^2}t\,. \\ \alpha(t) &= \frac{d\omega(t)}{dt} = 90\frac{1}{s^2}\,.\end{aligned}$$

2.2 Equations of Motions (Linear vs. Circular Motion)

Physically a linear movement and a circular movement are very different from each other, but mathematically both are governed by identical equations, thus we can use the same mathematical toolkit for solving them.

$$v = \frac{dx}{dt}\,, \qquad \omega = \frac{d\theta}{dt}\,. \tag{4}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}\,, \qquad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}\,. \tag{5}$$

For constant accelerations we get:

$$a(t) = const\,, \qquad \alpha(t) = const\,. \tag{6}$$

$$v(t) = at + v(t_0)\,, \qquad \omega(t) = \alpha t + \omega(t_0)\,. \tag{7}$$

$$x(t) = \frac{1}{2}at^2 + v(t_0)t + x_0\,, \qquad \theta(t) = \frac{1}{2}\alpha t^2 + \omega(t_0)t + \theta_0\,. \tag{8}$$

[4] We denote the angular acceleration by the greek letter α (alpha).

We also can eliminate the time in the last line by using the second line.

$$t = \frac{v(t) - v(t_0)}{a}\,, \qquad t = \frac{\omega(t) - \omega(t_0)}{\alpha}\,. \tag{9}$$

$$v(t)^2 = v(t_0)^2 + 2a\,[x(t) - x_0]\,, \qquad \omega(t)^2 = \omega(t_0)^2 + 2\alpha\,[\theta(t) - \theta_0]\,. \tag{10}$$

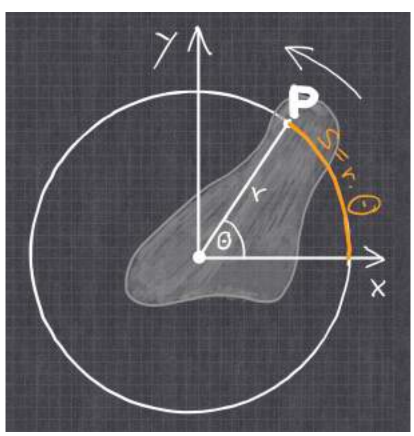
Example L1.4: Consider the circular saw from Example L1.2 with an emergency stop button. Pressing this button, the blade still does 2 turns plus an angular rotation of 59°. Assuming a constant angular acceleration, what is the numerical value of α?

$$\begin{aligned} \theta(t_0) &= 0\,, \qquad \theta(t) = \frac{2\cdot 360^\circ + 59^\circ}{360^\circ}2\pi = 13.5961\,, \\ \omega(t) &= 0s^{-1}\,, \qquad \omega(t_0) = 523.6s^{-1}\,, \\ \Rightarrow \alpha &= \frac{\omega(t)^2 - \omega(t_0)^2}{2[\theta(t) - \theta(t_0)]} = -\frac{(523.6s^{-1})^2}{2\cdot 13.5961} = -20164s^{-2}\,. \end{aligned}$$

2.3 Relating Angular with Linear Kinematics

Any point P of a rotating body has of course a linear speed and an acceleration. If P is in the distance r of the rotation axis, then we get for the arc length s

$$s = r\theta\,. \tag{11}$$



Thus the point P is moving with a velocity v

$$v = \frac{ds}{dt} \quad = \quad r\frac{d\theta}{dt} = r\omega\,. \tag{12}$$

Example L1.5: The blade from Example L1.2 has a diameter of 24 *inch*. What linear speed has the edge of the blade?

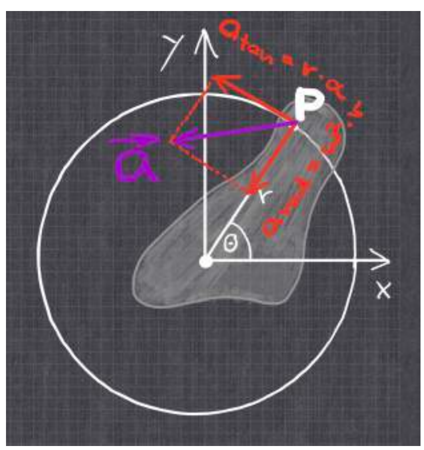
$$v = r\omega = \frac{1}{2} 24 \cdot 2.54 \cdot 10^{-2} m \cdot 523.6 s^{-1} = 160 \frac{m}{s}\,.$$

In Chapter 3.4. of the textbook we defined the tangential and radial component of the acceleration:

$$a_{tan} \equiv \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha\,, \tag{13}$$

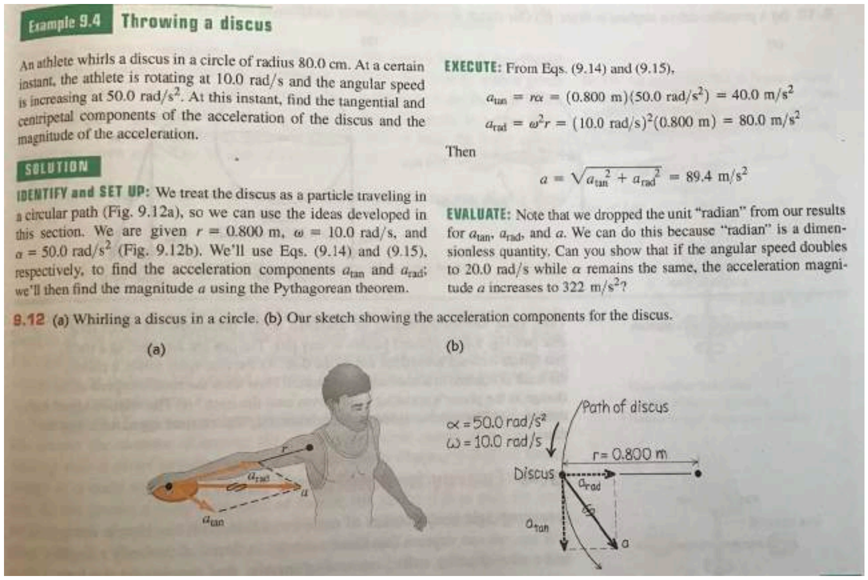
$$a_{perp} \equiv \frac{v^2}{r} = \omega^2 r\,. \tag{14}$$

With these two components at hand we can construct the linear acceleration vector $\vec{a}$[5].



[5]We follow the vector notation of the textbook. In the exam $\vec{a}$ will be denoted by $\underline{a}$.

Example L1.6: Acceleration when throwing a discus = Example 9.4 from textbook at page 287



Example 9.4 Throwing a discus

An athlete whirls a discus in a circle of radius 80.0 cm. At a certain instant, the athlete is rotating at 10.0 rad/s and the angular speed is increasing at 50.0 rad/s^2. At this instant, find the tangential and centripetal components of the acceleration of the discus and the magnitude of the acceleration.

SOLUTION

IDENTIFY and SET UP: We treat the discus as a particle traveling in a circular path (Fig. 9.12a), so we can use the ideas developed in this section. We are given $r = 0.800$ m, $\omega = 10.0$ rad/s, and $\alpha = 50.0$ rad/s^2 (Fig. 9.12b). We'll use Eqs. (9.14) and (9.15), respectively, to find the acceleration components a_{tan} and a_{rad}; we'll then find the magnitude a using the Pythagorean theorem.

EXECUTE: From Eqs. (9.14) and (9.15),

$$a_{tan} = r\alpha = (0.800\ \text{m})(50.0\ \text{rad/s}^2) = 40.0\ \text{m/s}^2$$

$$a_{rad} = \omega^2 r = (10.0\ \text{rad/s})^2(0.800\ \text{m}) = 80.0\ \text{m/s}^2$$

Then

$$a = \sqrt{a_{tan}^2 + a_{rad}^2} = 89.4\ \text{m/s}^2$$

EVALUATE: Note that we dropped the unit "radian" from our results for a_{tan}, a_{rad}, and a. We can do this because "radian" is a dimensionless quantity. Can you show that if the angular speed doubles to 20.0 rad/s while α remains the same, the acceleration magnitude a increases to 322 m/s^2?

9.12 (a) Whirling a discus in a circle. (b) Our sketch showing the acceleration components for the discus.

Please have a look at the textbook pages 288 - 294 before next lecture

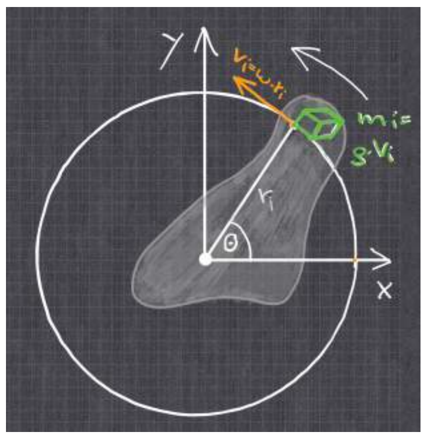
3 Lecture 2: Rotation of Rigid Bodies 2

Textbook pages 288- 294

3.1 Energy in Rotational Motion

Consider a rotating body (angular velocity ω) to consist of different particles with the masses m_1, m_2, ... at the distances r_1, r_2, ... from the rotation axis. Then the i-th particle has a kinetic energy of

$$E_{kin,i} = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i r_i^2 \omega^2 \,. \tag{15}$$



The total kinetic energy of the body reads then

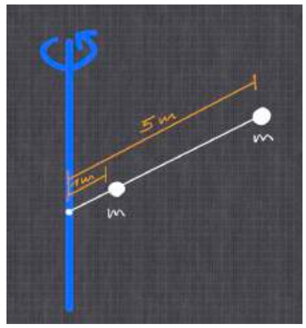
$$E_{kin} = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\omega^2 \sum_i m_i r_i^2 \,, \tag{16}$$

$$= \frac{1}{2} I \omega^2 \,, \tag{17}$$

with the **moment of inertia I**

$$I = \sum_i m_i r_i^2 \,. \tag{18}$$

Example L2.1: What is the kinetic energy of a body with a mass of 100 kg rotating at 1 turn per second at a distance of 1 m from the rotation axis?



How will the rotation energy change, if

a) the distance is increased to $5m$, while the angular velocity stays the same?

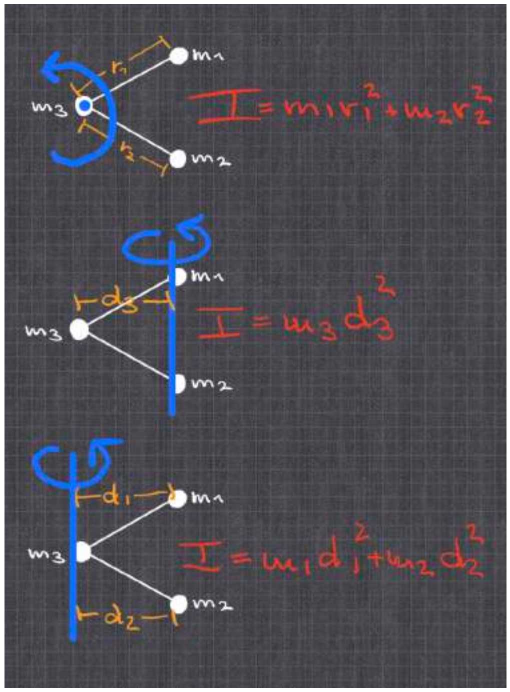
b) the angular velocity is increased by a factor of 5, while the distance of the rotation axis stays the same?

$$E = \frac{1}{2} m r^2 \omega^2 = \frac{1}{2} \cdot 100kg \cdot 1m^2 \cdot \left(\frac{2\pi}{1s}\right)^2 = 1974J \,.$$

$$a) \rightarrow 49348J \,.$$

$$b) \rightarrow 49348J \,.$$

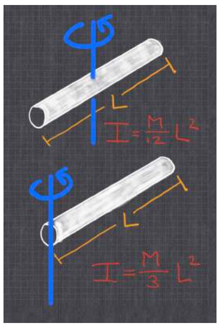
Example L2.2: Dependence of the moment of inertia from the rotation axis



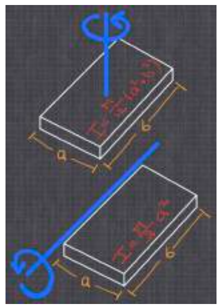
Remarks:

- The larger ω, the larger the kinetic energy
  - To get E_{kin} in Joule, ω has to be measured in radians/second.
- The larger I, the larger the kinetic energy
  - The larger the mass of the body, the larger the moment of inertia
  - The larger the mass is away from the rotation axis, the larger the moment of inertia
  - The size of I depends on the geometrical distribution of the mass

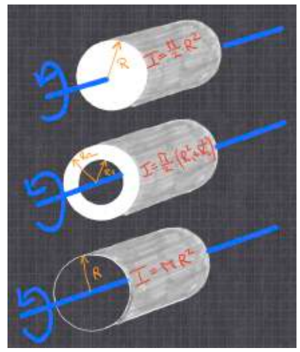
Moments of inertia of various bodies



Slender rods



Rectangular, thin plate



Cylinder

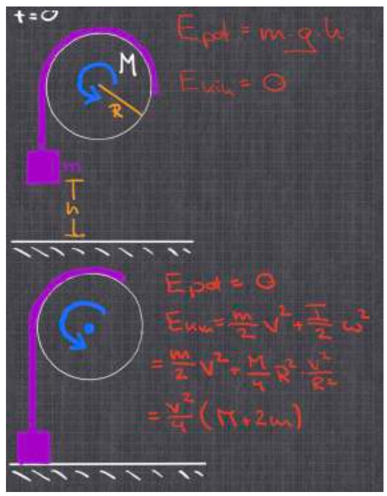
Sphere

Example L2.3: Determine the rotational energy of the Earth!
The radius of the Earth is 6371 km; we assume the Earth to be a solid sphere with constant density. The mass of the Earth is $5.972 \cdot 10^{24}\ kg$.

$$
\begin{aligned}
E_{kin} &= \frac{1}{2} I\omega^2 = \frac{1}{2}\frac{2}{5} MR^2 \left(\frac{2\pi}{T}\right)^2 \\
&= \frac{1}{5} \cdot 5.972 \cdot 10^{24} kg \cdot (6.371 \cdot 10^6 m)^2 \left(\frac{2\pi}{24 \cdot 3600 s}\right)^2 \\
&= 2.56 \cdot 10^{29} J\,.
\end{aligned}
$$

1 ton of TNT is equal to $4.2 \cdot 10^9 J$. Thus the rotation energy of the Earth is equal to $6.13 \cdot 10^{19}$ tons of TNT. The Hiroshima bomb was equivalent to 15 kilotons of TNT, hence the rotation energy of the Earth is equivalent to $4.09 \cdot 10^{15}$ Hiroshima bombs.

Example L2.4: Unwinding cable around a solid cylinder.
A light cable is wrapped around a solid cylinder with mass M and radius R. The cylinder can rotate frictionless around a stationary horizontal axis. To the free end of the cable a body with mass m is tied. At time $t = 0$ we release the body from the height h. What is the angular velocity of the cylinder, when the body hits the ground?



$$
\begin{aligned}
m \cdot g \cdot h &= \frac{v^2}{4}\left(M + 2m\right) , \\
v &= 2\sqrt{g \cdot h \frac{m}{M + 2m}} = \sqrt{2g \cdot h \frac{1}{1 + \frac{M}{2m}}} , \\
\omega &= \frac{v}{R} = \sqrt{2\frac{g \cdot h}{R^2}\frac{1}{1 + \frac{M}{2m}}} . \qquad (19)
\end{aligned}
$$

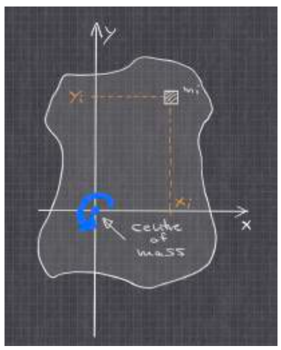
3.2 Parallel Axis Theorem

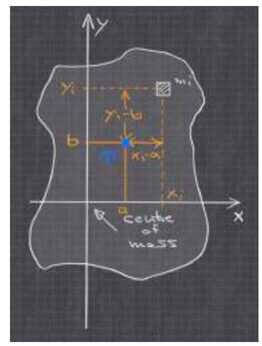
This theorem states that the moment of inertia for a body of mass M about an axis through the point P (in distance R from the centre of mass) I_P is given by

$$I_P = I_{C.M.} + MR^2 , \qquad (20)$$

with the moment of inertia for the same body of mass M about an axis through its center of mass $I_{C.M.}$.
Proof: $I_{C.M.}$ is given by

$$I_{C.M.} = \sum_i m_i(x_i^2 + y_i^2)\,. \tag{21}$$



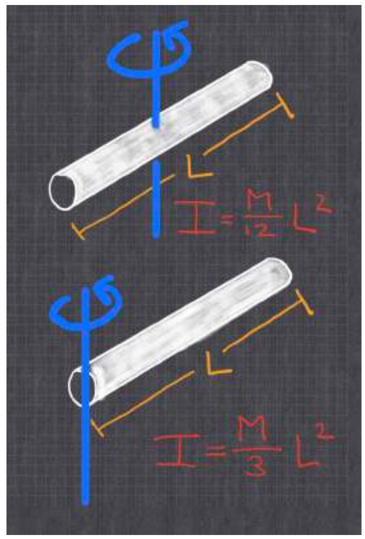


The moment of inertia of the same body about an axis through the point P is given by

$$\begin{aligned} I_P &= \sum_i m_i\left[(x_i - a)^2 + (y_i - b)^2\right] \\ &= \sum_i m_i\left[x_i^2 - 2x_i a + a^2 + y_i^2 - 2by_i + b^2\right] \\ &= \sum_i m_i(x_i^2 + y_i^2) - 2a\sum_i m_i x_i - 2b\sum_i m_i y_i + (a^2 + b^2)\sum_i m_i \\ &= I_{C.M.} + 0 + MR^2\,, \end{aligned} \tag{22}$$

with the mass of the body $M = \sum_i m_i$ and the distance $R = \sqrt{a^2 + b^2}$ of the point P from the centre of mass. Remember according to Section 8.5. of the textbook, the centre of mass is defined as $x_{C.M.} = \sum_i m_i x_i / M$ (similar for $y_{C.M.}$) and we have defined our coordinate system to have the centre of the mass in the origin, thus $\sum_i m_i x_i = 0 = \sum_i m_i y_i$.

Example L2.5: Use the parallel axis theorem to proof the second value of the moment of inertia, assuming the first value is correct.



$$I_2 = I_1 + M\left(\frac{L}{2}\right)^2 = \frac{M}{12}L^2 + \frac{M}{4}L^2 = \frac{M}{3}L^2\,.$$

Please have a look at the textbook pages 295 - 307

4 Lecture 3: Rotation of Rigid Bodies 3

Textbook pages 295 - 307

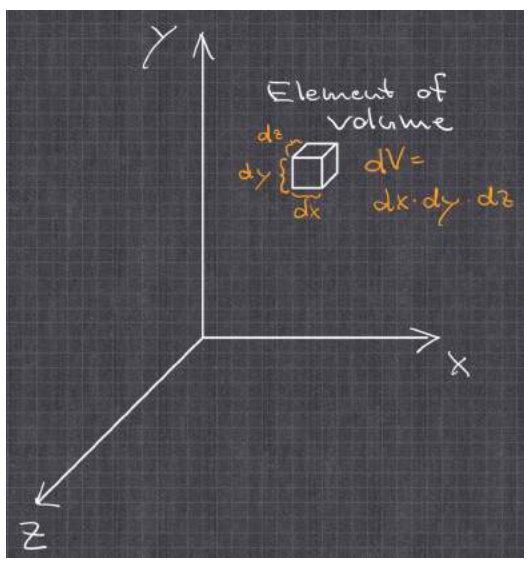
4.1 Calculation of Moments of Inertia

The **moment of inertia I** is defined as

$$I = \sum_i r_i^2 m_i = \int r^2 dm = \rho \int r^2 dV \,. \tag{23}$$

Many times cartesian coordinates (i.e. x, y and z) are not best suited for integrating over the volume of a body - in particular, if the body has some rotational symmetries.

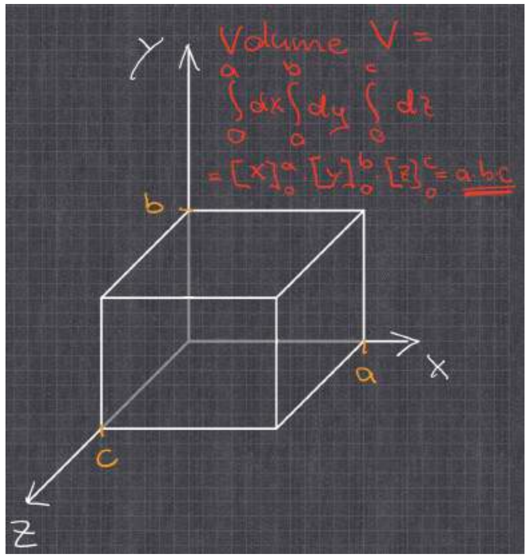
4.1.1 Cartesian Coordinates



In the well-known cartesian coordinates the element of volume dV is given by $dV = dx \cdot dy \cdot dz$. The volume of a body can be calculated according to

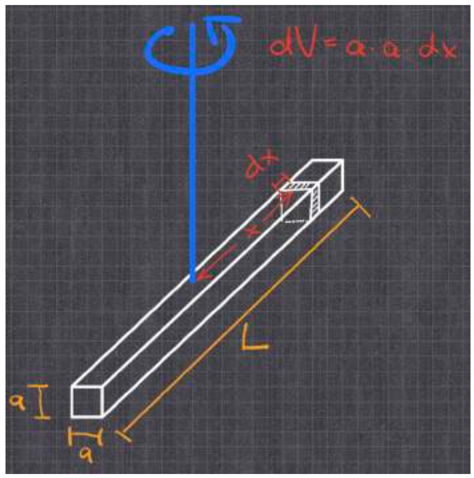
$$\text{Volume} = \int_{x_i}^{x_f} dx \int_{y_i}^{y_f} dy \int_{z_i}^{z_f} dz \,, \tag{24}$$

depending on its borders $x_i, x_f, \ldots$.
The volume of a cuboid can thus be determined as



$$V \;=\; \int_0^a dx \int_0^b dy \int_0^c dz = [x]_0^a \, [y]_0^b \, [z]_0^c = a \cdot b \cdot c \,. \tag{25}$$

Example L3.1 Calculate the moment of inertia of this piece of wood around the axis through the centre of mass in cartesian coordinates! Assume that the piece of wood is very thin, i.e. $a \ll L$.

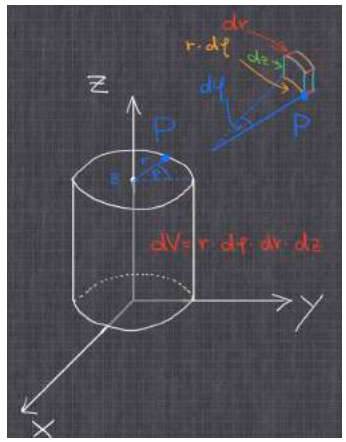


$$I = \rho \int r^2 dV = \rho \int_{-\frac{L}{2}}^{+\frac{L}{2}} x^2 a^2 dx = \rho \left[\frac{x^3}{3} \right]_{-\frac{L}{2}}^{+\frac{L}{2}} a^2 = \rho a^2 L \frac{L^2}{12} = \frac{M}{12} L^2 \,. \quad (26)$$

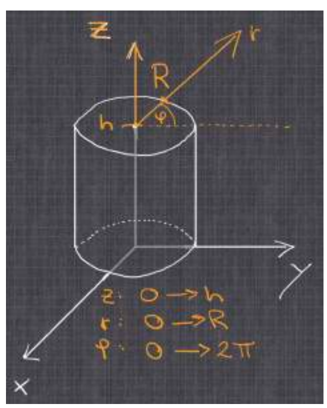
This coincides with the results for a slender rod, we were giving above.

4.1.2 Cylinder Coordinates

Here we use instead of (x, y, z) the coordinates (r, ϕ, z), where r is the distance from the z-axis and ϕ (=phi) is the rotational angle around the z-axis.



In these coordinates the volume element reads $dV = rd\phi \cdot dr \cdot dz$. The volume of a cylinder of height h and radius R can be calculated as



$$V = \int_0^h \int_0^{2\pi} \int_0^R r d\phi \cdot dr \cdot dz = \int_0^h dz \int_0^{2\pi} d\phi \int_0^R r dr$$

$$= \quad [z]_0^h\,[\phi]_0^{2\pi}\left[\frac{r^2}{2}\right]_0^R = R^2\pi h\,. \tag{27}$$

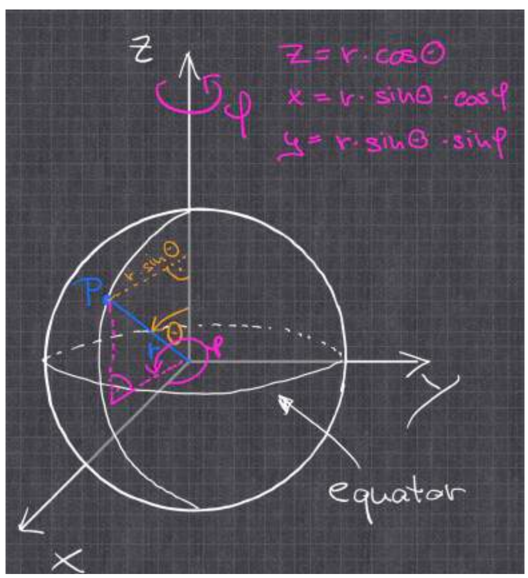
Example L3.2 Calculate the moment of inertia of a cylinder with radius R, length L and mass M around its symmetry axis.

$$\begin{aligned} I &= \rho\int r^2 dV = \rho\int_0^L dz\int_0^{2\pi} d\phi\int_0^R r^3 dr \\ &= \rho\,[z]_0^L\,[\phi]_0^{2\pi}\left[\frac{r^4}{4}\right]_0^R = \frac{1}{2}\cdot\rho R^2\pi L\cdot R^2 = \frac{M}{2}R^2\,. \end{aligned} \tag{28}$$

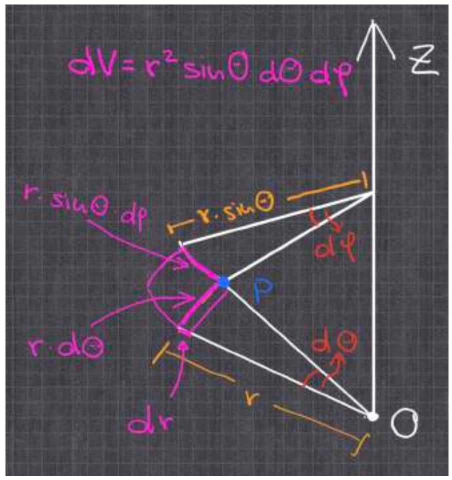
This coincides with the results for a solid cylinder, we were giving above.

4.1.3 Spherical Coordinates

Here we use instead of (x, y, z) the coordinates (r, ϕ, θ), where r is the distance from the origin, ϕ is the rotational angle around the z-axis and θ is the angle relative to the z-axis, with the North Pole sitting at $\theta = 0$ (and thus the equator at $\theta = \pi/2$).



In these coordinates the volume element reads $dv = rd\phi \cdot dr \cdot r\sin\theta d\theta$.



The volume of a sphere with radius R can be calculated as

$$\begin{aligned} V &= \int_0^R \int_0^{2\pi} \int_0^{\pi} r^2 dr d\phi \sin\theta d\theta = \int_0^R r^2 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \\ &= \left[\frac{r^3}{3}\right]_0^R [\phi]_0^{2\pi} [-\cos\theta]_0^{\pi} = \frac{R^3}{3} \cdot 2\pi \cdot 2 = \frac{4}{3} R^3 \pi \,. \end{aligned} \tag{29}$$

Why do we intergate θ only up to π and not 2π?

Example L3.3 Calculate the moment of inertia of a solid sphere with radius R about an axis going through the origin of the sphere. Here we have to be careful with our defining formulae

$$I = \rho \int r^2 dV \,. \tag{30}$$

When deriving it we denoted by r the distance from the rotation axis, while we denote with r the distance from the origin, when using spherical coordinates. In spherical coordinates the distance from the rotation axis is given as $r \sin\theta$. Thus we get

$$\begin{aligned} I &= \rho \int r^2 \sin^2\theta dV = \rho \int_0^R r^4 dr \int_0^{2\pi} d\phi \int_0^{\pi} \sin^3\theta d\theta \\ &= \rho \left[\frac{r^5}{5}\right]_0^R [\phi]_0^{2\pi} \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} \\ &= \rho \frac{R^5}{5} \cdot 2\pi \cdot (2 + \frac{1}{3}(-2)) = \frac{2}{5}R^2 \cdot \frac{4}{3}R^3 \pi\rho = \frac{2}{5}MR^2 \,. \end{aligned} \tag{31}$$

This coincides with the results for a solid sphere, we were giving above. Trick: $\sin^3\theta = \sin^2\theta \cdot \sin\theta = (1 - cos^2\theta)\sin\theta = \sin\theta - \cos^2\theta \cdot \sin\theta$. The derivative of $\cos^3\theta$ is $3\cos^2\theta \cdot (-\sin\theta)$.

Please have a look at the textbook pages 308 - 320 before next lecture

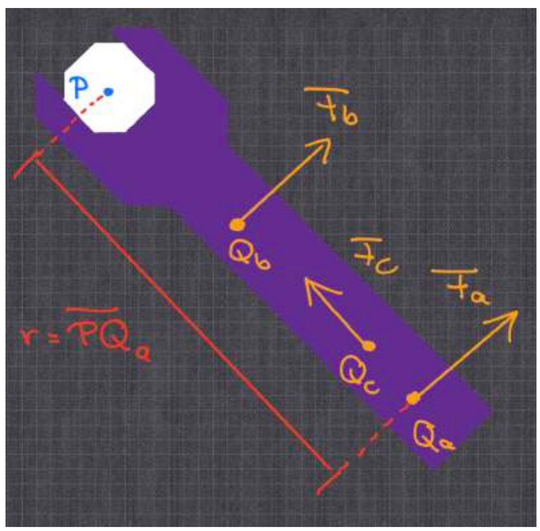
5 Lecture 4: Dynamics of Rotational Motion 1

Textbook pages 308 - 320

5.1 Torque

Well-known: Forces change translational motions.
How can a force change a rotational motion?

Example L4.1 Use a wrench to loosen a tight bolt - the leverage will be important!



The torque at point P exerted by the force $\vec{F}_a$ is given by

$$\tau = r \cdot |\vec{F}_a|\,. \tag{32}$$

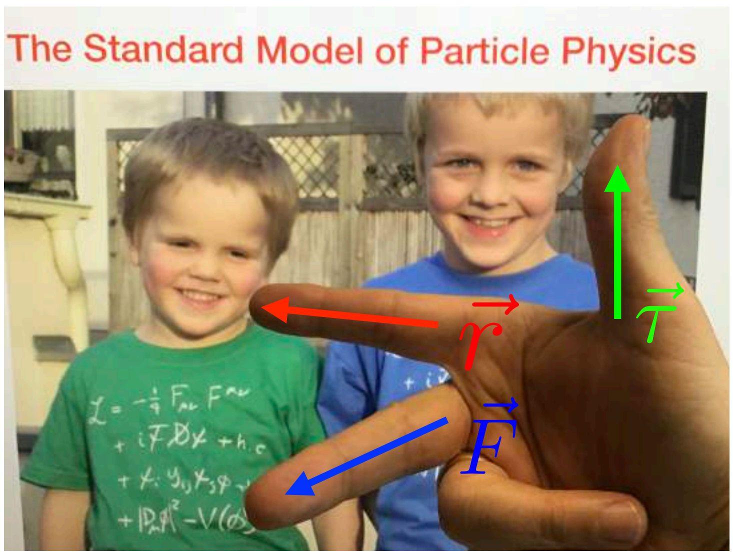
Remarks:

1. Torque is measured around a point P.
2. The larger the leverage (i.e. r), the larger the torque. $\vec{F}_a$ creates a larger torque than $\vec{F}_b$.

3. If the force is acting at the point Q, then only the component vertical to $r = \overline{PQ}$ is contributing to the torque ($\vec{F}_c$ creates no torque). The general definition of the torque is

$$\vec{\tau} = \vec{r} \times \vec{F}\,, \quad (33)$$
$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\phi\,. \quad (34)$$

The direction of the torque vector is given by the right hand rule:



4. Torque has the same units ($1\ Nm$) as energy ($1\ J = 1\ Nm$), but is a completely different physical concept.

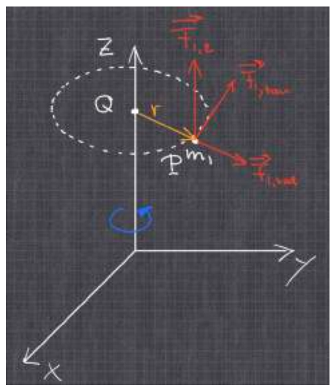
5.2 Torque and Angular Acceleration for a Rigid Body

In the same way as a force is leading to a linear acceleration, a torque will lead to an angular acceleration.
Consider a body of mass m_1 at point P that is rotating around the z-axis. Any arbitrary force $\vec{F}_1$ that is acting on the body, can be decomposed into three components

1. A z-component $\vec{F}_{1,z}$.

2. A radial component $\vec{F}_{1,rad}$.

3. A tangential component $\vec{F}_{1,tan}$ - this is the only component that will create a torque in the point Q.



The tangential force gives linear acceleration $a_{1,tan}$:

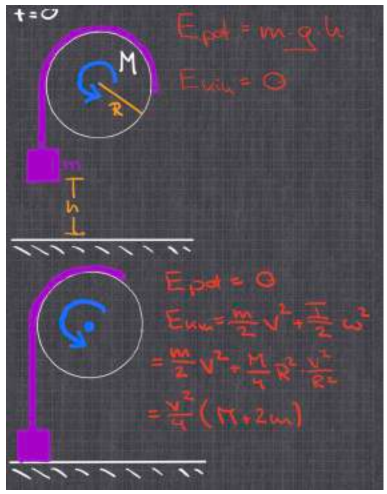
$$
\begin{aligned}
F_{1,tan} &= m_1 a_{1,tan} = m_1 r_1 \alpha_z \,, && (35)\\
r_1 F_{1,tan} &= m_1 r_1^2 \alpha_z \,, && (36)\\
\tau_1 &= m_1 r_1^2 \alpha_z = I_1 \alpha_z \,. && (37)
\end{aligned}
$$

Now we have a direct relation of the angular acceleration α_z with the causing torque τ_1. Next we consider an extended body to be built out of many small mass elements m_i in the distance r_i from the rotation axis, then we get (remember that for a rigid body all mass elements have the same angular acceleration)

$$\tau \equiv \sum_i \tau_i = \sum_i I_i \alpha_z = I \alpha_z \,. \qquad (38)$$

This is the equivalent to Newton's second law.

Example L4.2 Unwinding a cable
A light cable is wrapped around a solid cylinder with mass M and radius R. The cylinder can rotate frictionless around a stationary horizontal axis. To the free end of the cable a body with mass m is tied. At time $t = 0$ we release the body from the height h.



1. What is the acceleration of the falling body?

2. What is the tension in the cable?

ad 1) For the force in the vertical direction we get the difference between the gravitational force on m and the tension T of the cable:

$$\sum F_y = mg - T = ma_y\,.$$

The tension T is creating a torque on the cylinder

$$\sum \tau_z \;=\; RT = I\alpha_z = \frac{M}{2}R^2\alpha_z = \frac{M}{2}R^2 \cdot \frac{a_y}{R} \Rightarrow T = \frac{Ma_y}{2}\,.$$

Thus we get

$$mg - \frac{Ma_y}{2} = ma_y \Rightarrow a_y = \frac{g}{1+\frac{M}{2m}}\,.$$

ad b) For the cable tension we get

$$T \;=\; \frac{M}{2}\frac{g}{1+\frac{M}{2m}} = \frac{M}{2m}\frac{mg}{1+\frac{M}{2m}} = \frac{mg}{1+\frac{2m}{M}}\,. \qquad (39)$$

Test: The velocity when hitting the ground is given by

$$v^2 \;=\; 2ah = \frac{2gh}{1+\frac{M}{2m}}\,,$$

which agrees with our result in Exercise L2.4!

5.3 Rigid Body Rotation about a Moving Axis

What happens if the rotation axis is also moving?

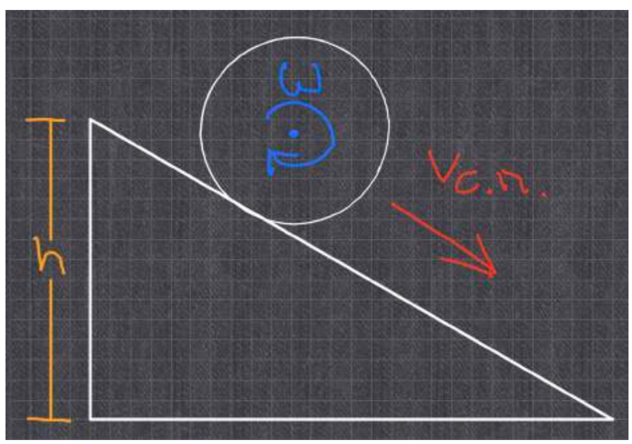
Theorem: Every possible motion of a rigid body can be represented as a combination of a translation of the centre of mass and a rotation around an axis through the centre of mass.

⇒ The total kinetic energy of a rigid body is given by the kinetic energy of its centre of mass plus the rotational energy around the centre of mass:

$$E_{kin} = \frac{M}{2} v_{C.M.}^2 + \frac{I_{C.M.}}{2} \omega^2 \,. \tag{40}$$

See proof on page 315 of the textbook.

Example: Rolling without slipping, i.e. $v_{C.M.} = \omega R$.



The dyanmics of a rigid body is thus given as

$$\sum \vec{F}_{ext.} = M\vec{a}_{C.M.} \,, \tag{41}$$

$$\sum \vec{\tau}_z = I_{C.M.}\alpha_z \,. \tag{42}$$

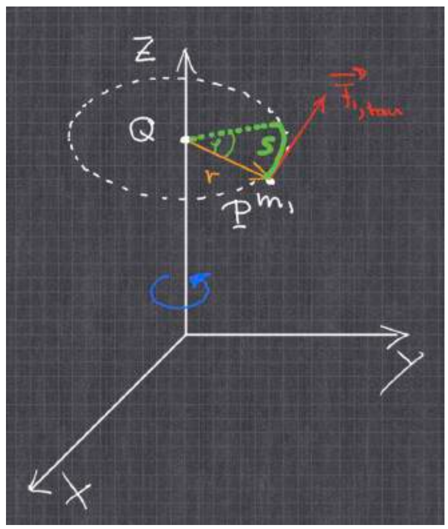
Please have a look at the textbook pages 320 - 343 before next lecture.

6 Lecture 5: Dynamics of Rotational Motion 2

Textbook pages 320 - 343

6.1 Work and Power in Rotational Motion

Consider the tangential force $\vec{F}_{1,tan}$ that is doing work on m_1 alongside s.



For an infinitesimal shift ds we get the following work:

$$dW = F_{tan} \cdot ds = F_{tan} \cdot r \cdot d\phi = \tau \cdot d\phi \,. \tag{43}$$

For the full length s we get then:

$$\begin{aligned} W &= \int_{\phi_i}^{\phi_f} \tau \cdot d\phi = \int_{\phi_i}^{\phi_f} I \cdot \alpha \cdot d\phi = \int_{\phi_i}^{\phi_f} I \cdot \frac{d\omega}{dt} \cdot d\phi \\ &= \int_{\omega_i}^{\omega_f} I \cdot \frac{d\phi}{dt} \cdot d\omega = \int_{\omega_i}^{\omega_f} I \cdot \omega \cdot d\omega = \frac{1}{2} I\omega_f^2 - \frac{1}{2} I\omega_i^2 \,, \end{aligned} \tag{44}$$

where we made a change of variables from ϕ to ω in the last line.
For the power we get

$$P = \frac{dW}{dt} = \tau \frac{d\phi}{dt} = \tau\omega\,. \quad (45)$$

6.2 Angular Momentum

The analogue of the force for a translational motion is the **torque** for a rotational motion.

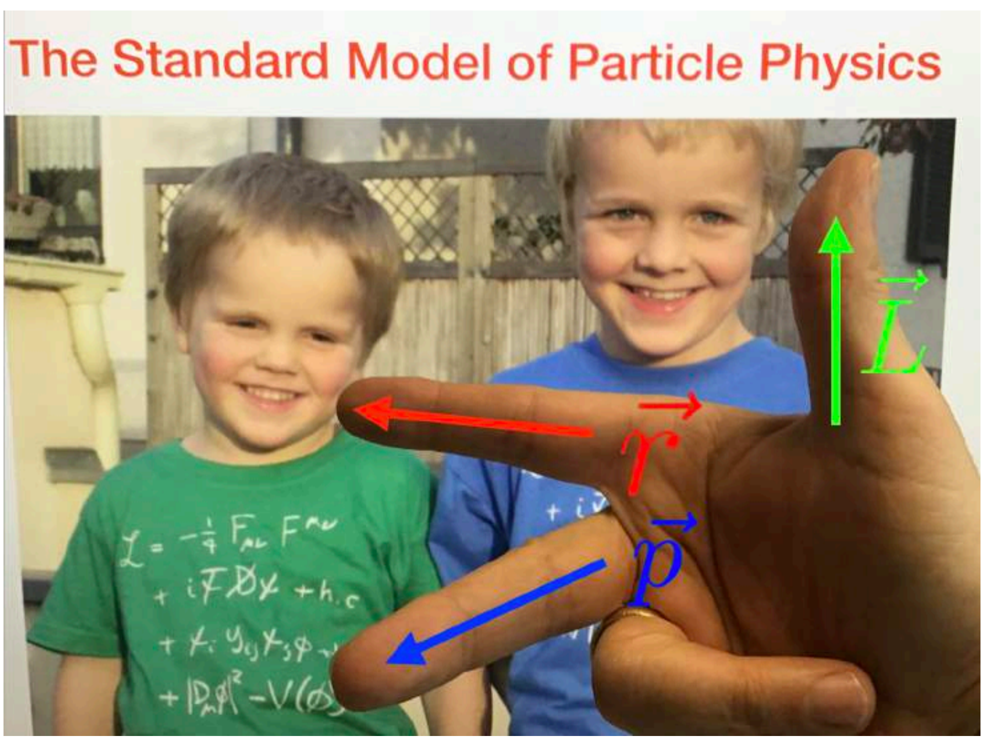
$$\vec{F} \Leftrightarrow \vec{\tau} = \vec{r} \times \vec{F}\,.$$

The analogue of the momentum for a translational motion is the **angular momentum** for a rotational motion.

$$\vec{p} \Leftrightarrow \vec{L} = \vec{r} \times \vec{p}\,, \quad (46)$$

$$m\vec{v} \Leftrightarrow \vec{L} = m\vec{r} \times \vec{v}\,. \quad (47)$$

The direction of $\vec{L}$ is given by the right hand rule:



For the time derivative of the angular momentum we get then:

$$\begin{aligned}\frac{d\vec{L}}{dt} &= \frac{d\vec{r}}{dt}\times m\vec{v} + \vec{r}\times m\frac{d\vec{v}}{dt}\\ &= \vec{v}\times m\vec{v} + \vec{r}\times m\vec{a} = \vec{r}\times\vec{F} = \vec{\tau}\,. \end{aligned} \tag{48}$$

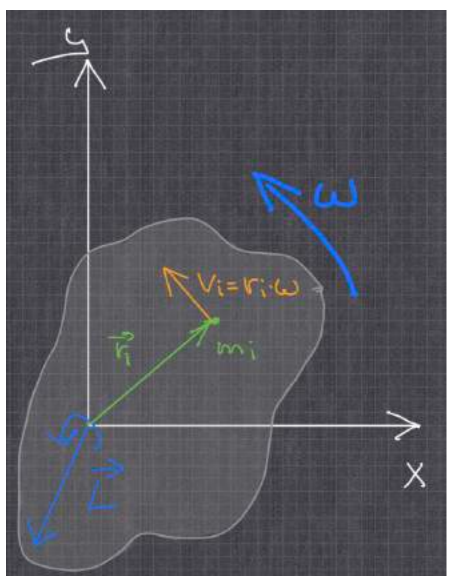
Compare this to Newton's 2nd law in the form of Eq.(8.4) in the text book

$$\frac{d\vec{p}}{dt} = \vec{F}\,. \tag{49}$$

All in all we get for the comparison of translational and rotational motions:

Momentum vs. angular momentum	$\vec{p}$	$\vec{L} = \vec{r}\times\vec{p}$
Force vs. torque	$\vec{F}$	$\vec{\tau} = \vec{r}\times\vec{F}$
Newtons 2nd law	$\frac{d\vec{p}}{dt} = \vec{F}$	$\frac{d\vec{L}}{dt} = \vec{\tau}$

We can further express the angular momentum in terms of the moment of inertia. To do so we consider first the angular momentum of a small piece (of mass m_i) of a rigid body that is rotating around the z-axis:



For the small mass element m_i we get:

$$L_i = r_i m_i v_i = r_i^2 m_i \omega = I_i \omega \,. \tag{50}$$

For the extended object in the $x-y$-plane we get

$$L = \sum_i L_i = \sum_i I_i \omega = I\omega \,. \tag{51}$$

We will use this definition in many cases!

6.3 Conservation of Angular Momentum

When the net external torque is zero, the total angular moment of the system is constant (conserved)!

$$\vec{\tau} = \vec{0} \Rightarrow \frac{d\vec{L}}{dt} = \vec{0} \,. \tag{52}$$

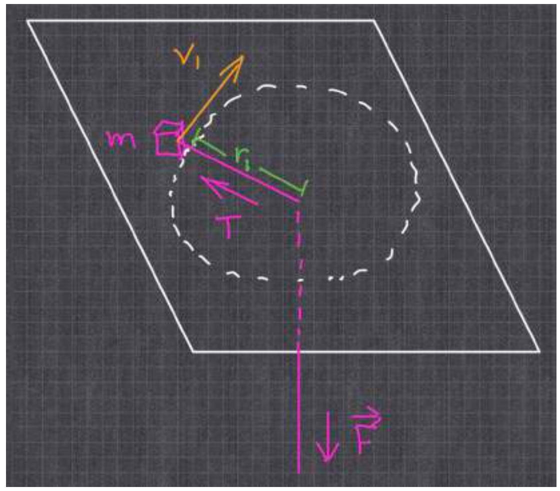
Example: Acrobats, Iceskater, Summersault,...:

$$L_1 = L_2 \,, \tag{53}$$

$$I_1\omega_1 = I_2\omega_2 \,. \tag{54}$$

Example L5.1 (10.42 from textbook):
An object of mass m is gliding frictionless in a circular motion over a plane. It is fixed by a rope, thus the radius r_1 is constant. The angular velocity is given by ω_1. Now we will pull with a force $\vec{F}$ on the loose end of the rope.

1. Question 1: Will the force $\vec{F}$ create a torque?
2. Question 2: What will be the angular velocity after the pull?
3. Question 3: What will be the kinetic energy after the pull?



1. Solution 1: The tension $\vec{T}$ creates the following torque $\vec{\tau} = \vec{r} \times \vec{T}$. Since $\vec{r}$ and $\vec{T}$ are parallel, the resulting torque is zero and the momentum is conserved, i.e. $L_1 = L_2$.

2. Angular momentum conservation gives

$$L_i = I_i\omega_i = m_i r_i^2 \omega_i \,, \tag{55}$$

$$L_1 = L_2 \Rightarrow \omega_2 = \omega_1 \left(\frac{r_1}{r_2}\right)^2 . \tag{56}$$

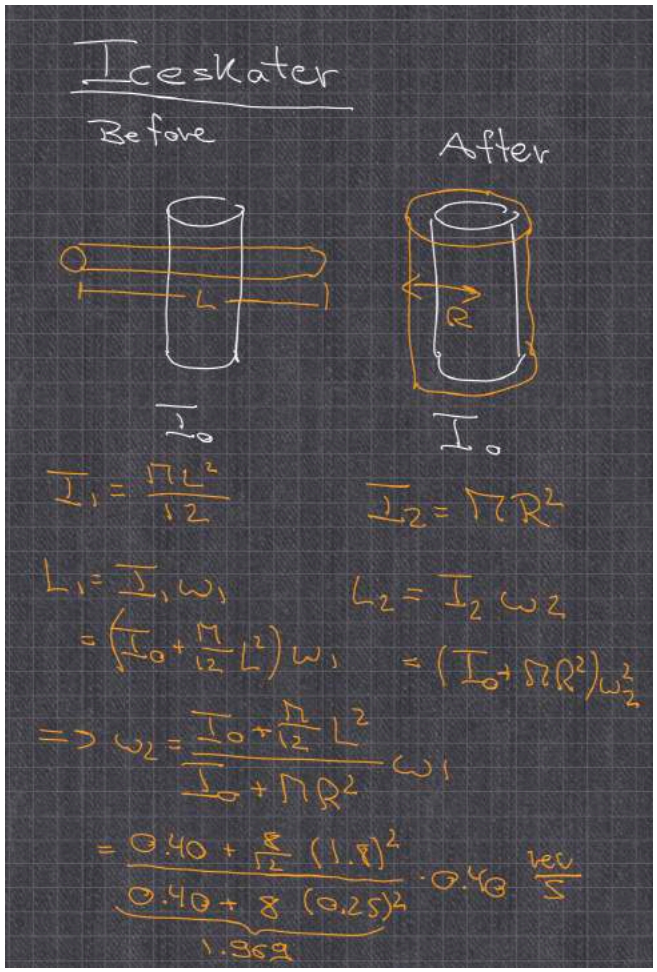
3. The kinetic energy is given by

$$E_{i,kin} = \frac{1}{2} I_i \omega_i = \frac{1}{2} m r_i^2 \omega_i^2 . \tag{57}$$

We can expresss the kinetic energy after the pull in terms of the kinetic energy before the pull:

$$\begin{aligned} E_{2,kin} &= \frac{1}{2} m r_2^2 \omega_2^2 = \frac{1}{2} m r_1^2 \left(\frac{r_2^2}{r_1^2}\right) \omega_1^2 \left(\frac{r_1}{r_2}\right)^4 \\ &= \frac{1}{2} m r_1^2 \omega_1^2 \left(\frac{r_1}{r_2}\right)^2 = E_{1,kin} \left(\frac{r_1}{r_2}\right)^2 . \end{aligned} \tag{58}$$

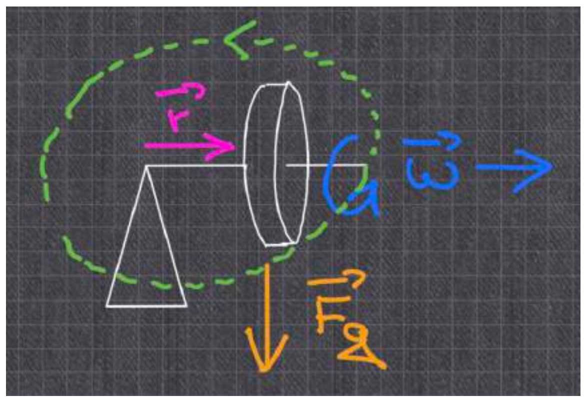
Example L5.2 (10.43 from textbook):
We also sketch here briefly the solution of Example 10.43 from the textbook:



6.4 Gyroscopes and Precession

Consider a flywheel that is rotating around its symmetry axis with the angular velocity ω. Next we put one end of the flywheel on a pivot - assume that the flywheel continues rotating. Thus the gravitational force F_g is acting on the flywheel and creates the torque

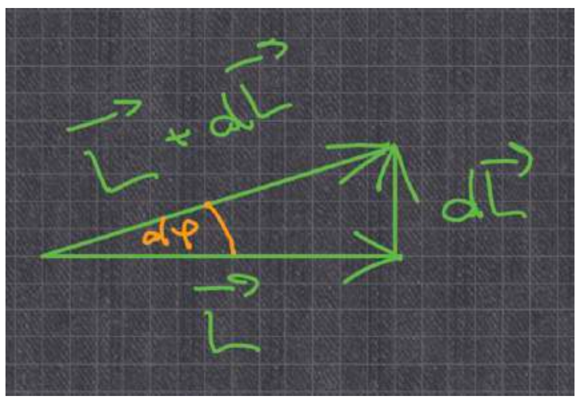
$$\vec{\tau} = \vec{r} \times \vec{F}_g \,. \tag{59}$$



The direction of the torque goes into the plane of this page and the torque creates the following change of momentum

$$d\vec{L} = \vec{\tau} dt \,. \tag{60}$$

Thus the flywheel starts to rotate into the plane of the page, this rotation is called **precession**. We can also estimate the angular velocity Ω of the precession, by looking at the infinitesimal change of the angular momentum:



Thus we get

$$\Omega \;=\; \frac{d\phi}{dt} = \frac{d\vec{L}/\vec{L}}{dt} = \frac{\vec{\tau}}{\vec{L}} = \frac{mgr}{I\omega}\,. \tag{61}$$

Please have a look at the textbook pages 344 - 372 before next lecture.

7 Lecture 6: Equilibrium and Elasticity 1

Textbook pages 344 - 372

7.1 Conditions for Equilibrium

Consider e.g. the stability of a building, a bridge,....
Remember: a rigid body is an idealisation, real bodies bend, stretch,...
For stability there should be not net force and not net torque, thus we get the following conditions for equilibrium:

1. **condition for equilibrium:**
   The sum of all external forces is zero.

$$\sum \vec{F} = \vec{0}\,, \quad (62)$$

$$\sum F_i = 0 \text{ for } i = x, y, z\,. \quad (63)$$

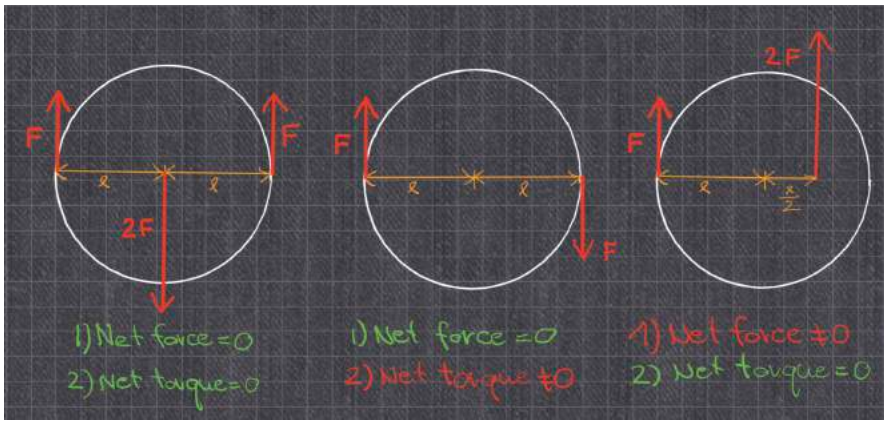
   Thus the centre of mass has zero acceleration.

2. **condition for equilibrium:**
   The sum of the torques due to all external forces is zero.

$$\sum \vec{\tau} = \vec{0}\,. \quad (64)$$

   Thus the body does not rotate.

Example L6.1:



7.2 Centre of Gravity

Remember: the centre of mass given by

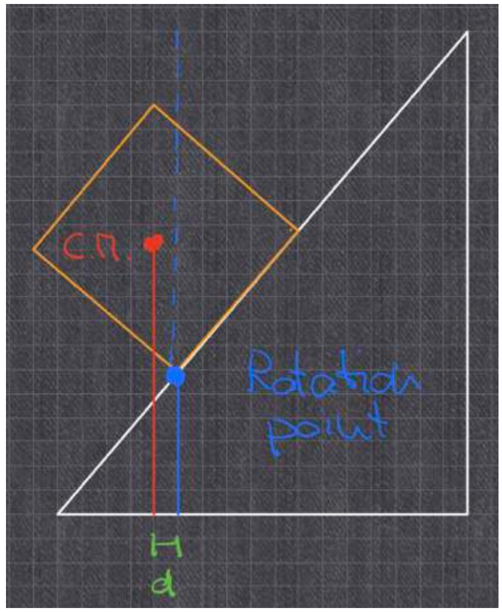
$$x_{C.M.} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i x_i}{\sum_i m_i}, \tag{65}$$

$$y_{C.M.} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i y_i}{\sum_i m_i}, \tag{66}$$

$$z_{C.M.} = \frac{m_1z_1 + m_2z_2 + m_3z_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i z_i}{\sum_i m_i}, \tag{67}$$

$$\Rightarrow \vec{r}_{C.M.} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \ldots}{m_1 + m_2 + m_3 + \ldots} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}. \tag{68}$$

Consider a block on an incline



Will it fall over?
Expectation: Yes, if the centre of mass if left of the axis.
Proof: the sum of all torques on the mass elements of an extended body is given by

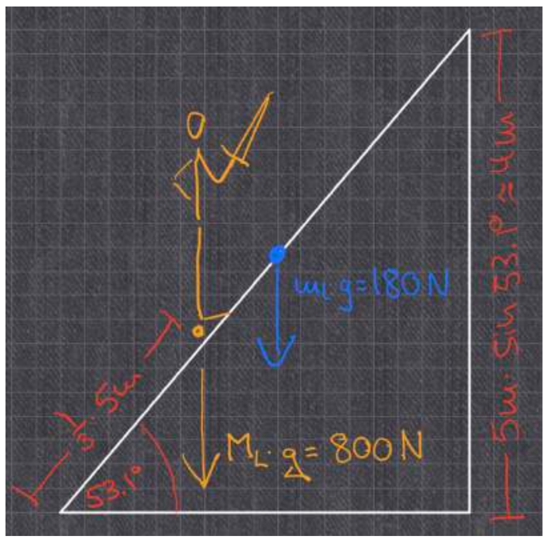
$$\vec{\tau} = \sum_i \vec{\tau}_i = \sum_i \vec{r}_i \times \vec{F}_{i,g} = \sum_i m_i \vec{r}_i \times \vec{g} \tag{69}$$

$$= \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} \sum_i m_i \times \vec{g} = \vec{r}_{C.M.} M \times \vec{g} = \vec{r}_{C.M.} \times \vec{F}_g \, . \tag{70}$$

The total torque is evaluated as if all the mass would sit in the centre of mass. In other wortds: the centre of gravity is identical to the centre of mass.
Be aware: The derivation assumed that $\vec{g}$ is the same for all mass elements - this might not be always the case!

7.3 Solving Rigid Body Equilibrium Problems

Example L 6.2 (11.3 from text book): Will the ladder slip?
Sir Lancelot ($M_L \cdot g = 800N$) climbs a ladder ($L = 5m$, $m_l \cdot g = 180N$) that leans against a wall with an angle of 53.1° against the horizontal. Lancelot pauses at one third of the way up the ladder.



a) Find the normal and friction forces at the base of the ladder.

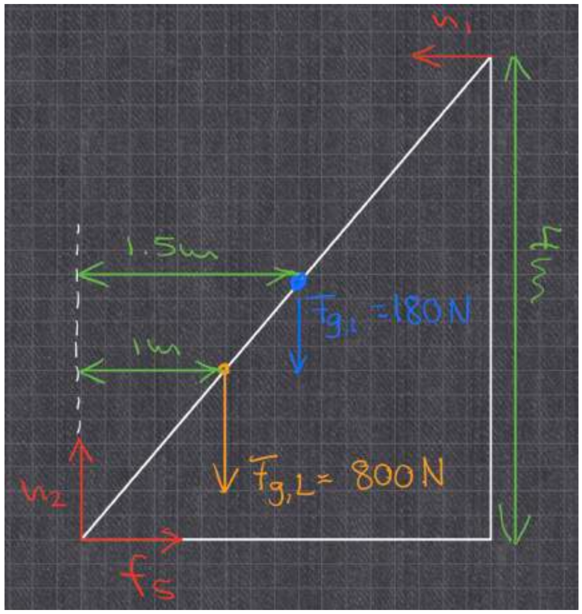
b) Find the minimum coefficient of static friction needed to prevent slipping.

c) Find the magnitude and direction of the contact force on the base.

Solution:

a) List of all forces

- Ladder: $180N$ at C.M.
- Lancelot: $800N$ at 1/3
- Wall: Normal force n_1
- Floor: Normal force n_2 and static friction force $f_s \leq \mu n_2$



Our equlibrium conditions give

$$0 = \sum F_x = f_s + (-n_1)\,. \tag{71}$$

$$0 = \sum F_y = n_2 + (-F_{l,g}) + (-F_{L,g})\,. \tag{72}$$

$$0 = \sum \tau_{Floor} = n_1 \cdot 4.0m - F_{l,g} \cdot 1.5m - F_{L,g} \cdot 1m\,. \tag{73}$$

This results in

$$n_2 = 980N\,, \qquad n_1 = 267.5N\,, \qquad f_s = 267.5N\,. \tag{74}$$

b) Minimum coefficient of static friction needed to prevent slipping.

$$\mu_{Min} = \frac{f_s}{n_2} = \frac{267.5}{980} = 0.27\,. \tag{75}$$

c) Find the magnitude and direction of the contact force on the base $\vec{F}_B$.

$$\vec{F}_B = \begin{pmatrix} f_s \\ n_2 \end{pmatrix} = \begin{pmatrix} 268\,N \\ 980\,N \end{pmatrix}. \tag{76}$$

$$\left|\vec{F}_B\right| = \sqrt{268^2 + 980^2}\,N = 1016\,N\,. \tag{77}$$

$$\theta = \arctan\left(\frac{980}{268}\right) = 1.30 = 75^\circ\,. \tag{78}$$

Remark: We had to assume that there is no friction on the wall - else the problem is not solvable with our equilibrium conditions.

Please have a look at the textbook pages 344 - 372 before next lecture

8 Lecture 7: Equilibrium and Elasticity 2

Textbook pages 344 - 372

8.1 Stress, Strain and Elastic Moduli

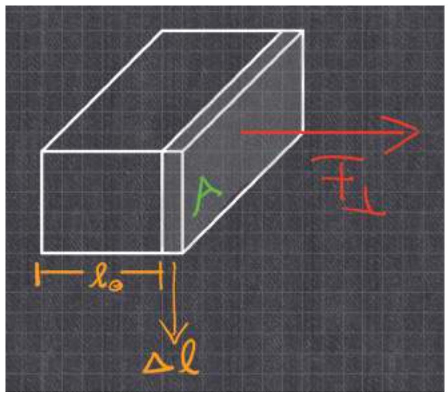
In a realistic body there are deformations and for each kind of deformation, we will introduce a quantity called **stress** characterising the force (per unit area) that causes the deformation.
Strain describes the resulting deformation.
For small values of stress and strain we find often that they are proportional

$$\frac{\text{Stress}}{\text{Strain}} = \text{Elastic} \quad \text{modulus} \quad (\text{Hooke's law})\,.$$

Remember: Hooke's (1635- 1703, contemporary of Newton) law for an ideal spring.

8.1.1 Tensile and Compressive Stress and Strain

Definition: Tensile Stress



$$\text{Tensile} \quad \text{stress} = \frac{F_\perp}{A} \,. \tag{79}$$

The unit of tensile stress is 1 Pascal $= 1Pa = 1N/m^2$.
Assume the tensile stress causes an elongation of the object from l_0 to $l = l_0 + \Delta l$.
Definition: Tensile Strain

$$\text{Tensile} \quad \text{strain} = \frac{l - l_0}{l_0} = \frac{\Delta l}{l_0} \,. \tag{80}$$

Definition: Young's modulus Y

$$Y = \frac{\text{Tensile} \quad \text{stress}}{\text{Tensile} \quad \text{strain}} = \frac{F_\perp l_0}{A \Delta l} \,. \tag{81}$$

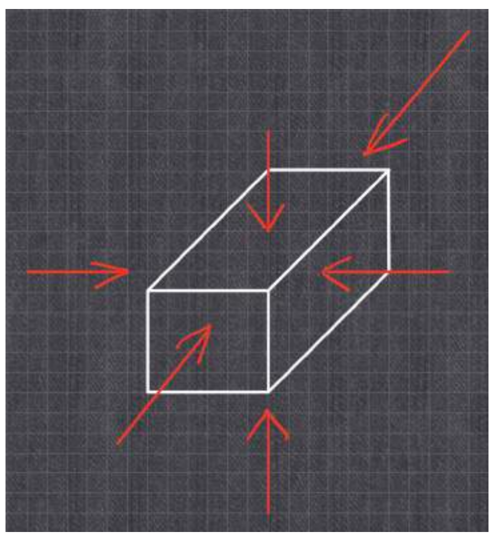
Metals have typical values of several times 10^{10} for Y. The larger Y, the lesser the material is stretchable.
Equivalently we can consider compressive stress; for many materials Y is identical for tensile and compressive stress - this holds e.g. not for concrete or stone, where $Y_{comp} \gg Y_{tens}$.

Example L7.1: A weight of $100kg$ hangs on an aluminium rod with an area of $0.30cm^2$. Determine the stress on the rod and the resulting strain and elongation.

$$\begin{aligned}
\text{Tensile stress} &= \frac{F_\perp}{A} = \frac{100kg \cdot 9.81N/kg}{0.30 \cdot 10^{-4}m^2} = 3.27 \cdot 10^7 Pa\,,\\
\text{Tensile strain} &= \frac{\text{Tensile stress}}{Y} = \frac{3.27 \cdot 10^7 Pa}{7.0 \cdot 10^{10} Pa} = 4.67 \cdot 10^{-4}\,,\\
\Delta l &= \text{Tensile strain} \cdot l = 4.67 \cdot 10^{-4} \cdot 2m = 1mm\,.
\end{aligned}$$

8.1.2 Bulk Stress and Strain

Definition: Bulk Stress/ Pressure
A uniform pressure form all sides creates a bulk stress or volume stress



$$\text{Pressure (Bulk stress)} = \frac{F_\perp}{A}\,. \tag{82}$$

The pressure in fluids increases with depth.
It causes a change in volume of the object from V_0 to $V = V_0 + \Delta V$.
Definition: Bulk volume Strain

$$\text{Bulk volume strain} = \frac{V - V_0}{V_0} = \frac{\Delta V}{V_0}\,. \tag{83}$$

A change of pressure from p to $p + \Delta p$ induces a bulk strain: **Definition: Bulk modulus** B

$$B = \frac{\text{Bulk} \quad \text{stress}}{\text{Bulk} \quad \text{volume} \quad \text{strain}} = -\frac{\Delta p V_0}{\Delta} \,. \tag{84}$$

We have minus sign, because an increase in pressure always reduces the volume - B is positive. For small changes in pressure, B is a constant for fluids or solids. For gas B depends on the initial pressure p_0. $1/B$ is called the compressibility.

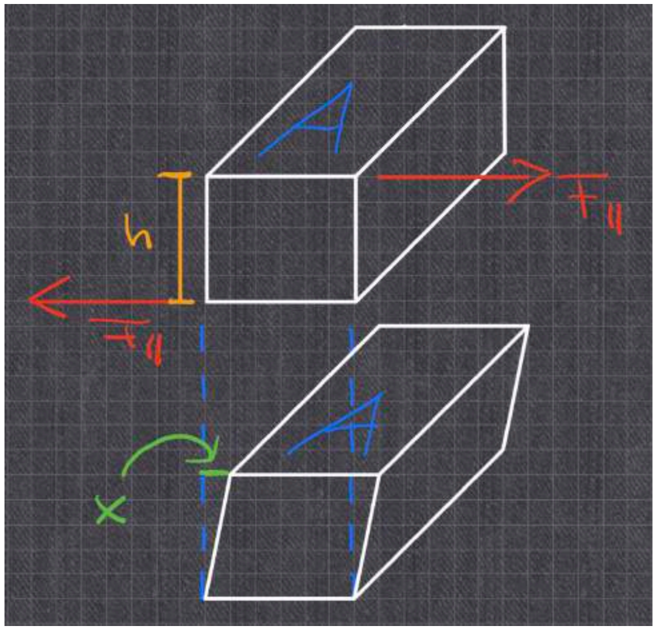
$$k = \frac{1}{B} = -\frac{1}{V_0}\frac{dV}{dp} \,. \tag{85}$$

Example L7.2: A hydraulic press contains $250l$ of oil. Find the change of volume if it is subjected to a pressure increase of $\Delta p = 1.6 \cdot 10^7 Pa$. The compressibility of oil is about $k = 2 \cdot 10^{-10} 1/Pa$.

$$dV = -kV_0 \Delta P = -2 \cdot 10^{-10} 1/Pa 1.6 \cdot 10^7 Pa 250l = -0.8l \tag{86}$$

8.1.3 Shear Stress and Strain

Definition: Shear Stress



$$\text{Shear} \quad \text{stress} = \frac{F_{\parallel}}{A} \,. \tag{87}$$

Assume the shear stress causes a shift of x

Definition: Shear Strain

$$\text{Shear} \quad \text{strain} = \frac{x}{h} \,. \tag{88}$$

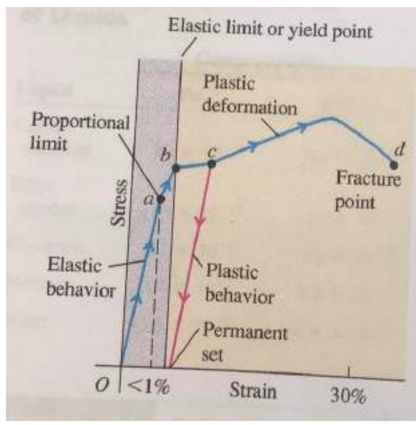
Definition: Shear modulus S

$$S = \frac{\text{Shear} \quad \text{stress}}{\text{Shear} \quad \text{strain}} = \frac{F_{\parallel} h}{A \Delta x} \,. \tag{89}$$

The values of S are many times in the range of $1/3Y ... 1/2Y$.

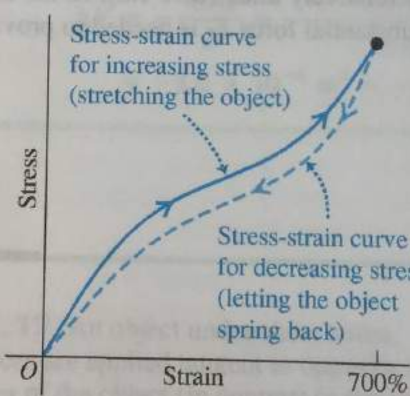
8.2 Elasticity and Plasticity

When is Hooke's law: $F = -kx$ applicable?



- up to point a: Hooke
- up to point b: reversible
- break apart at point d

11.19 Typical stress-strain diagram for vulcanized rubber. The curves are different for increasing and decreasing stress, a phenomenon called elastic hysteresis.



9 Lecture 8: Fluid Mechanics 1

Textbook pages 373 - 401

9.1 Density

$$\rho = \frac{M}{V} \tag{90}$$

List of densities:

$$\text{Air} \qquad \rho = 1.2\frac{kg}{m^3}\,, \tag{91}$$

$$\text{Water} \qquad \rho = 1.00 \cdot 10^3\frac{kg}{m^3}\,, \tag{92}$$

$$\text{Steel} \qquad \rho = 7.8 \cdot 10^3\frac{kg}{m^3}\,, \tag{93}$$

$$\text{Gold} \qquad \rho = 19.3 \cdot 10^3\frac{kg}{m^3}\,, \tag{94}$$

$$\text{White dwarf} \qquad \rho = 10^{10}\frac{kg}{m^3}\,, \tag{95}$$

$$\text{Neutron star} \qquad \rho = 10^{18}\frac{kg}{m^3}\,, \tag{96}$$

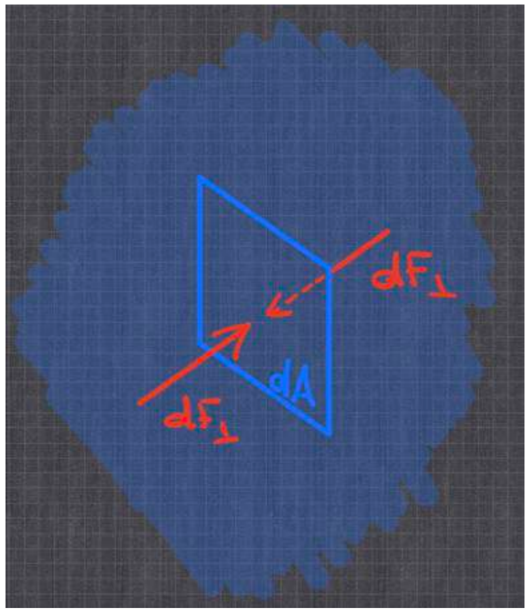
$$\text{Fat} \qquad \rho = 940\frac{kg}{m^3}\,, \tag{97}$$

$$\text{Bone} \qquad \rho = 1700...2500\frac{kg}{m^3}\,. \tag{98}$$

Definition: Specific gravity: $= \rho/\rho_{water}$.

9.2 Pressure in a Fluid

Inagine an imaginary surface dA within a fluid. Due to the movement of the molecules a force $F_\perp$ is acting on both sides of dA, with the same magnitude, but different direction.

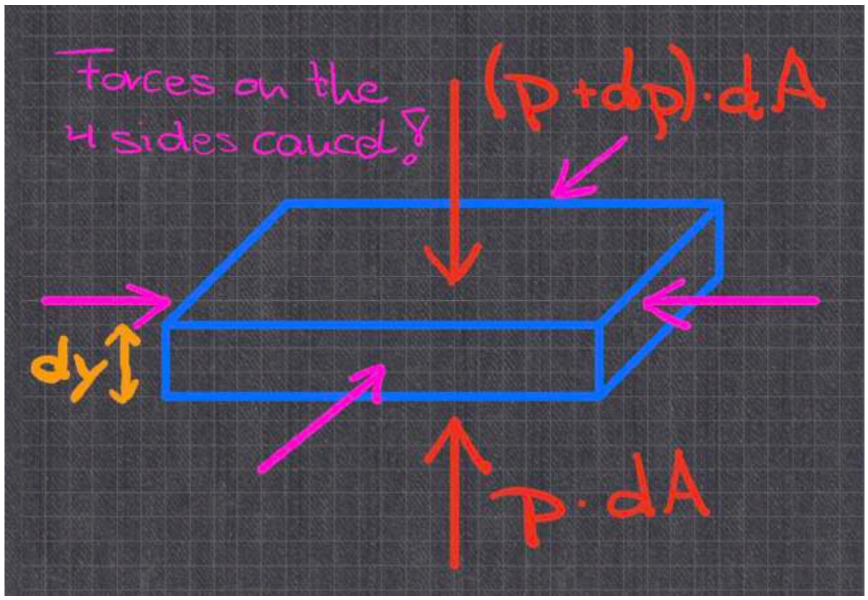


$$p(x) = \frac{dF_\perp(x)}{dA}\,. \tag{99}$$

If the pressure is the same at all points of a finite area A, then we get

$$p = \frac{F_\perp}{A}\,. \tag{100}$$

If the weight of the fluid is negligible, then the pressure is constant for the whole volume.
If the weight is not negligible: **Pascal's law**
Assume the density is constant (incompressible fluid).
Consider a small volume element of volume $dV = A \cdot dy$ and mass $dm = \rho \cdot dV$ (weight $dw = \rho \cdot g \cdot dV$).
The pressure at height y is p and at height $y + dy$ it is $p + dp$.



Since the fluid is in equlibrium we get

$$\begin{aligned} 0 &= \sum F_y = pA - (p + dp)A - \rho g A dy & (101) \\ \Rightarrow \frac{dp}{dy} &= -\rho g & (102) \\ \Rightarrow p_2 - p_1 &= -\rho g \left[y_2 - y_1 \right] . & (103) \end{aligned}$$

If p_0 is the pressure at the surface and $h = y_2 - y_1$ is the depth, then we get

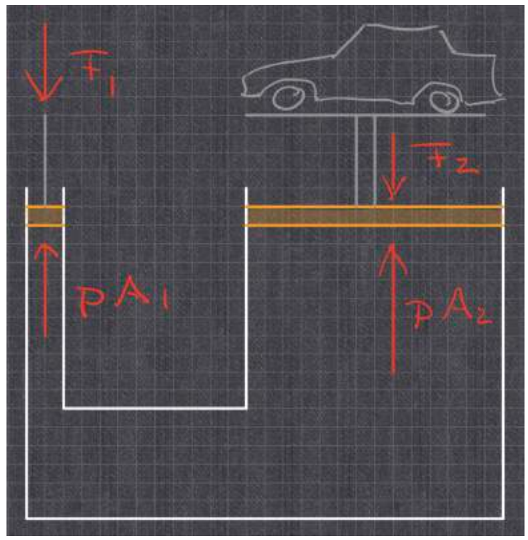
$$p = p_0 + \rho g h \,. \qquad (104)$$

Example L8.1 : What is the pressure at the bottom of a $12m$ deep water tank whose top is open to the atmosphere?
Remark: atmospheric pressure varies of course with the weather. Normal atmospheric pressure at sea level is defined as $1\,atmosphere(atm) = 101325Pa$.

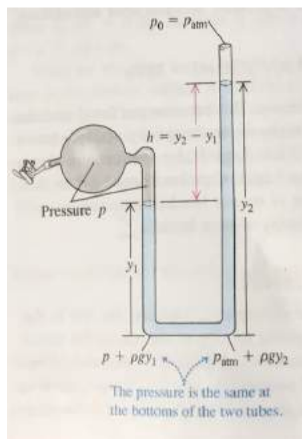
$$\begin{aligned} p &= p_0 + \rho g h \\ &= 1.01 \cdot 10^5 Pa + 1000\frac{kg}{m^3} 9.81\frac{N}{kg} 12m \\ &= 2.19 \cdot 10^5 Pa = 2.16atm\,. \end{aligned}$$

A car tyre with $p = p_0$ is flat, thus many times only the difference from p_0 matters.
Definition: Gauge pressure $= p - p_0$.
Eq.(104 tells us
If we increase p_0 then this increase will be transmitted undiminished to every point in the fluid (Pascal's law)
This simple observation has quite some dramatic consequencs, i.e. hydraulic lifts:
The pressure applied at point 1 is equal to the preassure at point 2

$$p = \frac{F_1}{A_1} = \frac{F_2}{A_2} \quad \Rightarrow \quad F_2 = \frac{A_2}{A_1} F_1 \tag{105}$$



For gases the assumption of constant density will hold only for very small heights, while liquids are more or less inompressible.
Pressure Gauges: e.g. open manometer



$$
\begin{aligned}
p + \rho g y_1 &= p_o + \rho g y_2 && (106) \\
p - p_{atm} &= \rho g (y_2 - y_1) = \rho g h \,. && (107)
\end{aligned}
$$

9.3 Buoyancy

Archimedes's principle: When a body is immersed in a fluid, the fluid exerts an upward force on the body equal to the weigth of the fluid displaced by the body.

Example L8.2: 150 kg golden statue in water

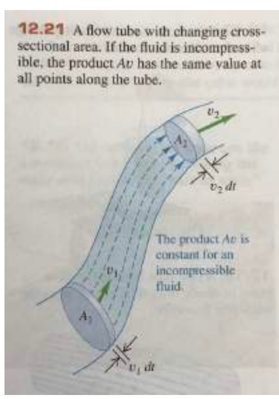
$$
\begin{aligned}
F_{g,gold} &= 150kg9.81\frac{N}{Kg} = 1471.5N \\
V_{gold} &= \frac{m_{gold}}{\rho_{gold}} = \frac{150kgm^3}{19300kg} = 0.00777202m^3 \\
F_{g,water} = m_{water}g &= V_{water}\rho_{water}g \\
&= V_{gold}\rho_{water}g = 0.00777202m^3 1000\frac{kg}{m^3}9.81 \\
&= 76.2435N \\
\Rightarrow F &= F_{g,gold} - F_{g,water} = 1395N
\end{aligned}
$$

10 Lecture 9: Fluid Mechanics 2

Textbook pages 373 - 401

10.1 Fluid Flow

Motion of fluids: **ideal fluid** = incompressible (i.e. $\rho = const$) and no internal friction (= **viscosity**). Many liquids are to a good incompressible.
Flow line = path of an individual particle in a moving fluid.
Steady flow = overall flow pattern does not change with time; every element passing through a certain point follows the same flow line.
Streamline = curve, whoose tangent is in the direction of the fluid velocity.
For a **steady flow**: streamline = flow line.
Flow tube: flow lines passing through an imiaginary area A from a flow tube. In **steady flow** flow tubes do not cross each other!
laminar flow vs **turbulent flow**



Continuity Equation: the mass of a moving fluid does not change with time

$$\begin{aligned} m_1 &= m_2\,, & (108)\\ \rho V_1 &= \rho V_2\,, & (109)\\ \rho A_1 dr_1 &= \rho A_2 dr_2\,, & (110)\\ \rho A_1 v_1 dt &= \rho A_2 v_2 dt\,, & (111)\\ \Rightarrow A_1 v_1 &= A_2 v_2\,. & (112) \end{aligned}$$

The continuity equation states that the product of area A times the flow velocity v is constant in a steady flow. **Volume flow rate**

$$\begin{aligned} dV &= A v dt\,, & (113)\\ \frac{dV}{dt} &= A v\,. & (114) \end{aligned}$$

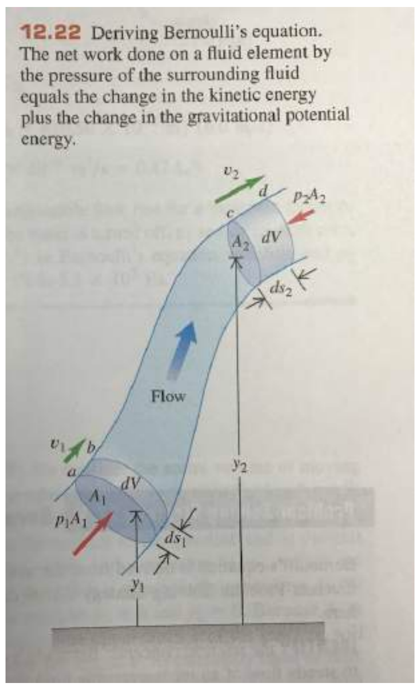
Continuity equation: volume flow rate = constant
"Still waters run deep" : v small, A large

For a compressible fluid the continuity equation reads

$$\rho_1 A_1 v_1 \quad = \quad \rho_2 A_2 v_2 \,. \tag{115}$$

10.2 Bernoulli's Equation

Continuity equation: variation of speed along the flow
Pascal: pressure depends on height
Bernoulli's equation: relates pressure, flow speed and heigth for the flow of an ideal imcompressible fluid.
Plumbing systems, hydroelectric generating stations, flight of airplanes



At time t the considered fluid element is in between the areas given by a and c, at time $t + \Delta t$ it is between b and d. Since the fluid is incompressible, we have

$$A_1 ds_1 = dV = A_2 ds_2 \,. \tag{116}$$

The work done on the fluid in the time dt is a sum of potential and kinetic energy.

$$dW \quad = \quad dE_{pot} + dE_{kin} \,, \tag{117}$$

$$dW = F_1 ds_1 + F_2 ds_2 = p_1 A_1 ds_1 - p_2 A_2 ds_2 \,. \tag{118}$$

The mechanical energy of the fluid in between b and c does not change!
At time t the fluid between a and b has the kinetic energy $\frac{1}{2}\rho(A_1 ds_1)v_1^2$.
At time $t + dt$ the fluid between c and d has the kinetic energy $\frac{1}{2}\rho(A_2 ds_2)v_2^2$.

$$dE_{kin} = \frac{1}{2}m_2 v_2^2 - \frac{1}{2}m_1 v_1^2 = \frac{1}{2}\rho dV(v_2^2 - v_1^2)\,. \tag{119}$$

At time t the fluid between a and b has the potential energy $\frac{1}{2}\rho(A_1 ds_1)gy_1$.
At time $t+dt$ the fluid between c and d has the potential energy $\frac{1}{2}\rho(A_2 ds_2)gy_2$.

$$dE_{pot} = mg(y_2 - y_1) = \rho dv(y_2 - y_1)\,. \tag{120}$$

Putting everything together

$$\Rightarrow (p_1 - p_2)dV = \frac{1}{2}\rho dV(v_2^2 - v_1^2) + \rho dV g(y_2 - y_1)\,, \tag{121}$$

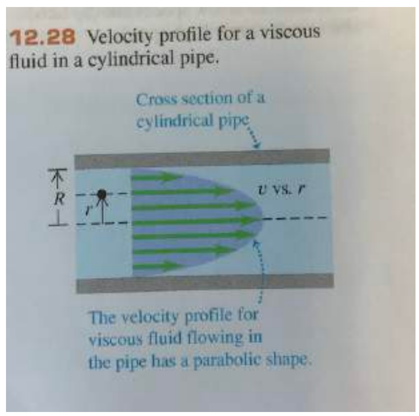
$$(p_1 - p_2) = \frac{1}{2}\rho(v_2^2 - v_1^2) + \rho g(y_2 - y_1)\,. \tag{122}$$

This is **Bernoulli's equation**: work done on a unit volume of fluid is equal to changes in kinetic and potential energy during the flow. A more convenient form:

$$p + \rho gy + \frac{1}{2}\rho v^2 = const\,. \tag{123}$$

10.3 Viscosity and Turbulence

Visosity = internal friction
Water has a low viscosity
Honey has a high viscosity
Flow in a pipe: velocity at the wall = 0, velocity in the centre is maximal



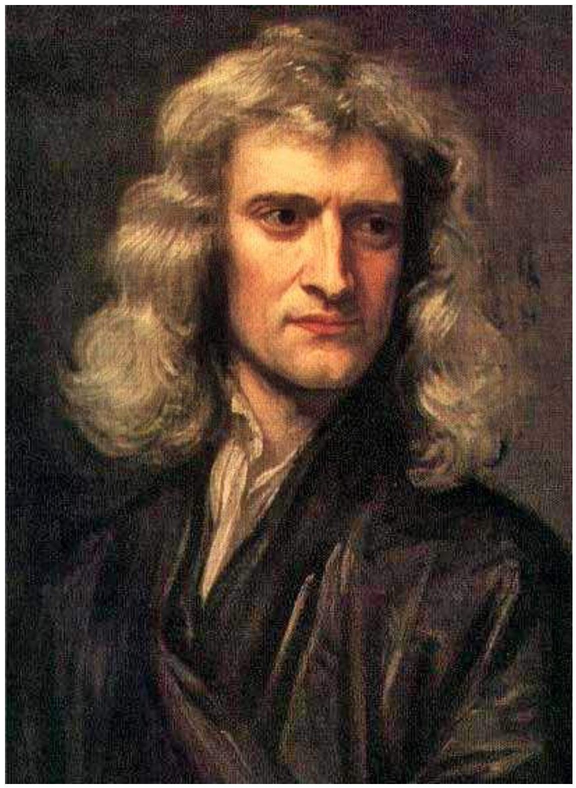
12.28 Velocity profile for a viscous fluid in a cylindrical pipe.

Turbulence: if the velocity of the fluid get larger than a critical value, the flow will no longer be laminar, but turbulent. E.g. picture of smoke. Bernoulli's equation is not applicable.
The greater the viscosity, the less probable is turbulence.

11 Lecture 10: Gravitation 1

Textbook pages 402 - 436

11.1 Newton's Law of Gravitation



Isaac Newton - 1642 - 1727

The gravitational attraction of two objects with masses m_1 and m_2, which are separated by the distance r is given as

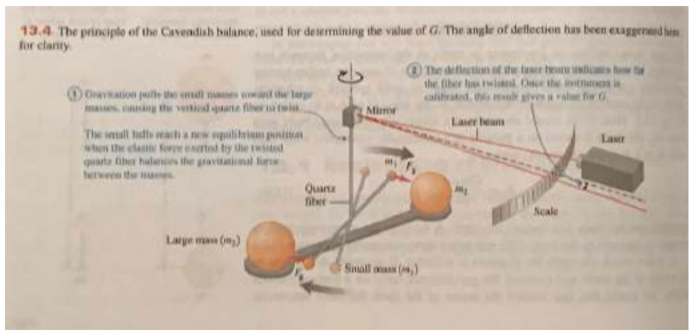
$$F = \frac{Gm_1 m_2}{r^2}\,. \tag{124}$$

Remarks:

a) G is the graviational constant, it is measured to be

$$G = 6.67408 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \,. \tag{125}$$

It is measured by a torsion balance (Cavendish):



b) Gravitational attraction of an extended, spherical symmetric body of mass M is identical to the gravitational attraction of a point-like particle with the same mass.

c) For an object of mass m on the surface of the Earth, we get

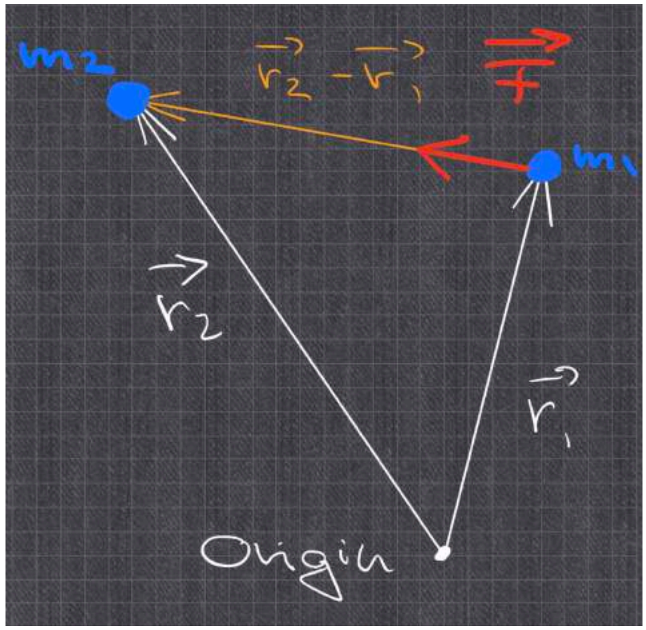
$$\begin{aligned} F &= \frac{G m M_{Earth}}{R_{Earth}^2} && (126) \\ &= \frac{6.67408 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \cdot 5.972 \cdot 10^{24} kg}{(6371 km)^2} m && (127) \\ &= 9.82 \frac{m}{s^2} \cdot m = gm \,. && (128) \end{aligned}$$

d) The gravitational law is universal, i.e. it holds for apples falling from trees, for planets orbiting the sun,...

e) Gravitation is always positive, even for anti-particles.

f) Direction of the force is given in the vector notation:

$$\vec{F} = \frac{G m_1 m_2}{|\vec{r}_2 - \vec{r}_1|^3} (\vec{r}_2 - \vec{r}_1) \,. \tag{129}$$



g) Gravitational forces combine vectorially

$$\vec{F}_{combined} = \vec{F}_1 = \vec{F}_2 \, . \tag{130}$$

h) It was formulated in Newton's work *Philosophiae Naturalis Principia Mathematica* ("the Principia"), first published on 5 July 1686. When Newton's book was presented in 1686 to the Royal Society, Robert Hooke made a claim that Newton had obtained the inverse square law from him.

i) Gravity is one of the 4 known fundamental forces in nature:

1. Gravity: infinite range.
2. Electro-magnetic force

$$F_{electric} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2} \, . \tag{131}$$

also infinite range.

3. Strong force: binds nucleons to nuclei and quarks to nucleons, acts only up to $10^{-15}m$.

4. Weak force: radioactive decay, energy production in the sun, acts only up to $10^{-18}m$.

j) Gravity is by far the weakest known force: the electric force between two protons is a factor of 10^{36} stronger than the gravitational force.

$$\frac{F_{electric}}{F_{gravity}} = \frac{1}{4\pi\epsilon_0 G}\frac{q_1 q_2}{m_1 m_2} \quad (132)$$

$$= \frac{1}{4\pi 8.854187817 \cdot 10^{-12}C^2N^{-1}m^{-2} \cdot 6.67408 \cdot 10^{-11}m^3kg^{-1}s^{-2}} \cdot \frac{(1.6021766208(98) \cdot 10^{-19}C)^2}{(1.672621898(21) \cdot 10^{-27}kg)^2} \quad (133)$$

$$= 1.23559 \cdot 10^{36}\frac{Ns^2}{mkg} = 1.23559 \cdot 10^{36}\,. \quad (134)$$

If we consider electrons instead of protons (the mass of the electron is a factor of 1836 smaller) then we get for the ratio

$$\frac{F_{electric}}{F_{gravity}} = (1836)^2 \cdot 1.23559 \cdot 10^{36} = 4.16505 \cdot 10^{42}\,. \quad (135)$$

But macroscopic bodies are electrically neutral, while gravitational effects are always summing up.
In astrophysics one has many times gravity as the only relevant force.

11.2 Weight

Definition: The weigth of a body is the total gravitational force exerted on the body by all other bodies in the universe.

Near the surface of the Earth, this is dominated by the Earth's effect, the same holds near the surface of other planets or the moon.

$$g_{Earth} = \frac{GM_{Earth}}{R_{Earth}^2} \quad (136)$$

$$= \frac{6.67408 \cdot 10^{-11}m^3kg^{-1}s^{-2} \cdot 5.972 \cdot 10^{24}kg}{(6371km)^2} \quad (137)$$

$$= 9.82\frac{m}{s^2}\,. \quad (138)$$

$$g_{Moon} = \frac{GM_{Moon}}{R^2_{Moon}} \quad (139)$$

$$= \frac{6.67408 \cdot 10^{-11} m^3 kg^{-1} s^{-2} \cdot 7.34767309 \cdot 10^{22} kg}{(1737km)^2} \quad (140)$$

$$= 1.625 \frac{m}{s^2} \approx \frac{1}{6} g_{Earth} \,. \quad (141)$$

We also have to take effects from the rotation of the Earth into account - this will slightly reduce the value of g_{Earth}.

How good is our approximate formulae $F = mg$?

$$F = \frac{GmM_{Earth}}{(R_{Earth} + h)^2} = \frac{GmM_{Earth}}{R^2_{Earth}} \frac{1}{\left(1 + \frac{h}{R_{Earth}}\right)^2} \quad (142)$$

$$\approx mg \left(1 - 2\frac{h}{R_{Earth}}\right) \,. \quad (143)$$

h	$1m$	$10m$	$100m$	$1000m$	$10000m$
$1 - 2\frac{h}{R_{Earth}}$	0.9999996860	0.999997	0.999969	0.999686	0.996861

11.3 Gravitational Potential Energy

Assuming a constant gravitational force $F = mg$, we obtained for the potential energy $E_{pot} = mgh$. How will the potential energy a body of the mass m in the gravity field of the Earth, look like for Newton's formula of gravitation?

$$E_{pot} = \int_{r_1}^{r_2} F_G dr = \int_{r_1}^{r_2} \frac{GM_E m}{r^2} \quad (144)$$

$$= \left[-\frac{GM_E m}{r} \right]_{r_1}^{r_2} = -GM_E m \left(\frac{1}{r_2} - \frac{1}{r_1} \right) \,. \quad (145)$$

We can check, whether our old formula is a good approximation of the new one by considering $r_2 = R_E + h$ and $r_1 = R_E$:

$$E_{pot} = -GM_E m \left(\frac{1}{R_E + h} - \frac{1}{R_E} \right) = -\frac{GM_E}{R_E} m \left(\frac{1}{1 + \frac{h}{R_E}} - 1 \right) \quad (146)$$

$$\approx -\frac{GM_E}{R_E} m \left(1 - \frac{h}{R_E} - 1 \right) = \frac{GM_E}{R^2_E} mh = mgh \,. \quad (147)$$

This is the well-known expression for the potential engery - thus everything is consistent.

Example L10.1: Escape velocity from Earth.

$$E_{kin,1} + E_{pot,1} = E_{kin,2} + E_{pot,2} \tag{148}$$

$$\frac{1}{2}mv^2 - \frac{GmM_E}{R_E} = 0 - \frac{GmM_E}{\infty} \tag{149}$$

$$\Rightarrow v = \sqrt{\frac{2GM_E}{R_E}} = 1.12 \cdot 10^4 \frac{m}{s} = 40200 \frac{km}{h} . \tag{150}$$

From the sun ($M_\odot = 1.99 \cdot 10^{30} kg,\ R = 6.96 \cdot 10^8 m$) we get $v = 6.18 \cdot 10^5 \frac{m}{s}$. With

$$M_{star} = \frac{4}{3} R_{star}^3 \pi \rho_{star} \tag{151}$$

we get

$$v_{escape} = \sqrt{\frac{2GM_{star}}{R_{star}}} = \sqrt{\frac{8\pi G \rho_{stsar}}{3}} R_{star} . \tag{152}$$

When will $v_{escape} = c$?

1. Stars, 500 times as big as sun and same density....

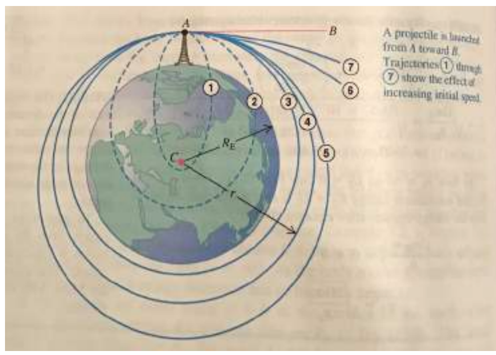
2. $$R_{star} = \frac{2GM_{star}}{c^2} . \tag{153}$$

   tihs is the **Schwarzschildradius** - for the sun this would be 2.956 km.

11.4 The Motion of Satellites

11.4.1 General Considerations

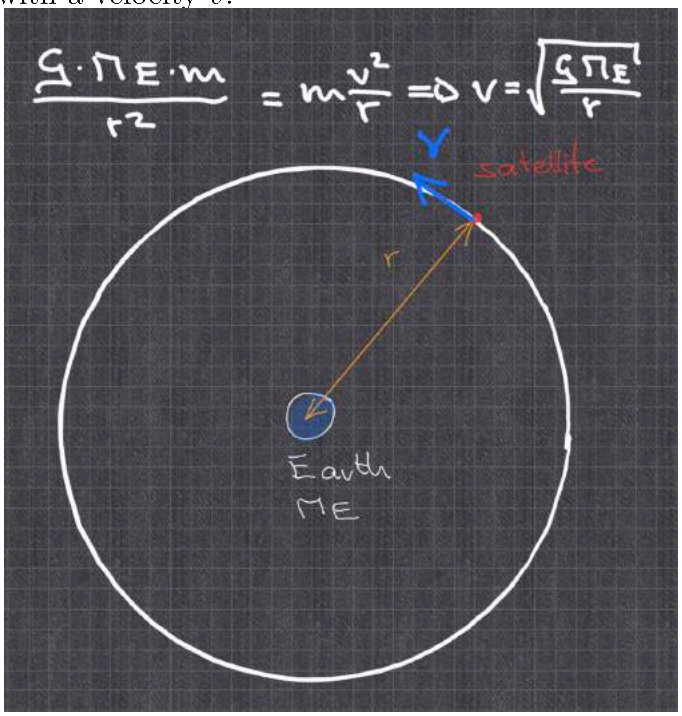
Imagine shooting a projectile from a tower and steadily increasing the initial velocity:



1-5: closed orbits are ellipses, a circle (4) is a special case of an ellipse
6-7: open orbits

11.4.2 Circular Orbits

Consider an object (satellite) of the mass m circulating around the Earth in a distance r with a velocity v:



$$\begin{aligned} F_{gravity} &= ma_{rad} && (154) \\ \frac{GM_E m}{r^2} &= m \cdot \frac{v^2}{r} && (155) \\ \Rightarrow v &= \sqrt{\frac{GM_E}{r}}\,. && (156) \end{aligned}$$

For a given radius r the velocity v is fixed.
The velocity does not depend on the mass of the satellite.
Period of evolution T

$$\begin{aligned} v &= \frac{2\pi r}{T} && (157) \\ \Rightarrow T &= \frac{2\pi r}{v} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM_E}}\,. && (158) \end{aligned}$$

Kepler's third law is a generalisation of this formula.

Example L10.2: Compare the Moon ($r = 384.000km$) and the ISS (International Space Station) ($r = 6371km + 429km$):

$$v_{Moon} = 1.0\frac{km}{s}\,, \qquad T_{Moon} = 27.3\,days\,, \tag{159}$$

$$v_{ISS} = 7.7\frac{km}{s}\,, \qquad T_{ISS} = 93\,minutes\,. \tag{160}$$

Energy in a circular orbit

$$E = E_{kin.} + E_{pot} = \frac{m}{2}v^2 - \frac{GM_E m}{r} \tag{161}$$

$$= \frac{GM_E m}{2r} - \frac{GM_E m}{r} = -\frac{GM_E m}{2r}\,. \tag{162}$$

This is equal to half of the potential energy.

Example L10.3 (13.6.): $1000kg$ satellite in $300km$ orbit

$$v = 7720\frac{m}{s}\,, \tag{163}$$

$$T = 90.6\,minutes\,, \tag{164}$$

$$a_{rad} = 8.92\frac{m}{s^2}\,. \tag{165}$$

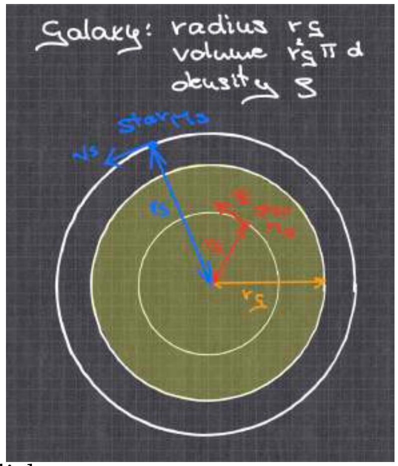
Energy, work

$$E_2 = -\frac{GmM_E}{2r} = -2.98 \cdot 10^{10} J\,, \tag{166}$$

$$E_1 = -\frac{GmM_E}{R_E} = -6.24 \cdot 10^{10} J\,, \tag{167}$$

$$W = E_2 - E_1 = 3.26 \cdot 10^{10} J\,. \tag{168}$$

Example L10.4: galaxy rotation curves as an indication for dark mater
Assume the galaxy can be described as a cylindric disk with radius r_G, heigth d and density ρ_0.

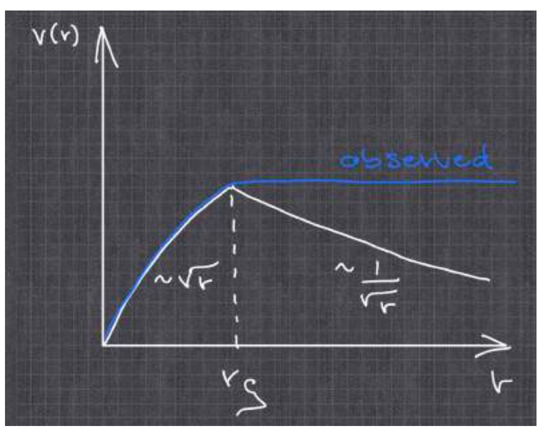


For a star outside the disk we get

$$v = \sqrt{\frac{G\pi r_G^2 d\rho}{r}} \propto \frac{1}{\sqrt{r}} \tag{169}$$

For a star inside the disk we get

$$v = \sqrt{\frac{G\pi r^2 d\rho}{r}} \propto \sqrt{r} \tag{170}$$



We observe for stars outside the disk, however, a constant behaviour or even a rise, but not a drop-off, which could be explained by some additional, invisible matter that is interacting via gravity - so-called **Dark Matter**.

12 Lecture 11: Gravitation 2

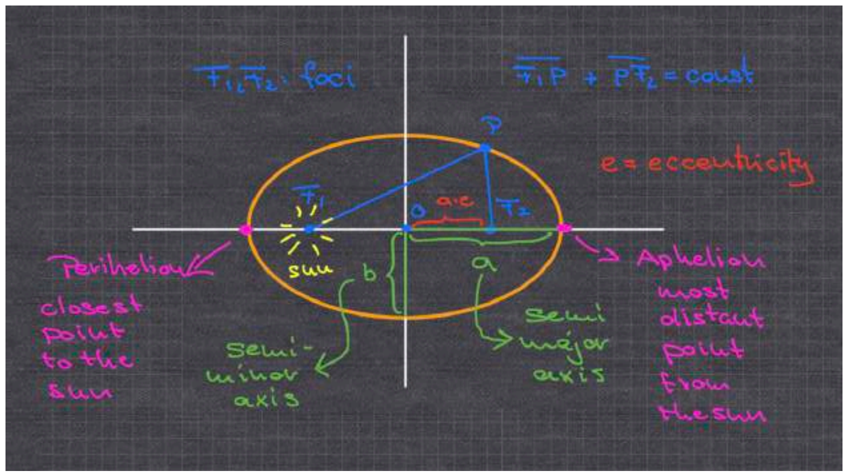
Textbook pages 402 - 436

Johannes Kepler 1571 - 1630

12.1 Kepler's Laws and the Motion of Planets

Kepler used an extensive data set from Tycho Brahe to discover three laws:

1. Each planet moves in an elliptical orbit, with the sun in the focus of the ellipse.



   If $e = 0$ then the ellipse is a circle; (Venus $e = 0.007$, Earth $e = 0.017$, Mercury $e = 0.206$).
   Newton has shown that a $1/r^2$ force leads to ellipses as closed orbits and parabola and hyperbola as open orbits.
   Predicting he perihelion precission of Mercury correctly was one of the biggest successes of Einstein's General Theory of Relativity.

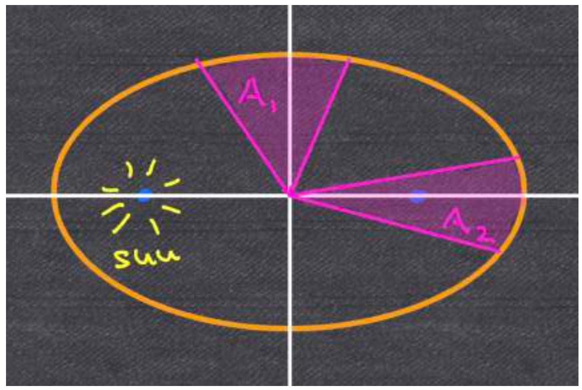
2. A line from the sun to a given planet sweeps out equal areas in equal times. This can also be stated as

$$\frac{dA}{dt} = \frac{\frac{1}{2} \cdot r \cdot r d\theta}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \text{const.} \tag{171}$$

$$= \frac{1}{2} r v \sin\phi = \frac{1}{2m} r p \sin\phi = \frac{L}{2m}\,, \tag{172}$$

   with ϕ being the angle between $\vec{r}$ and $\vec{v}$. Thus Kepler's second law is equal to the conservation of momentum.
   Momentum conservation holds for all central forces $\vec{F} = \text{factor} \cdot \vec{r}$:

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \text{factor} \cdot \vec{r} \times \vec{r} = 0\,. \tag{173}$$



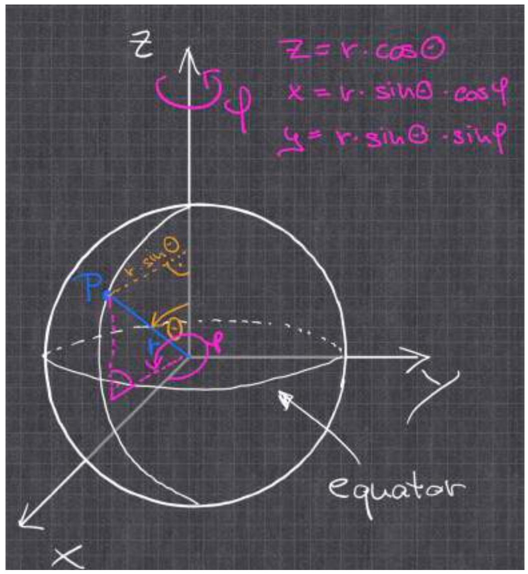
3. The periods of the planets squared are proportional to the third power of the major axis length of their orbit.

$$T = \frac{2\pi r}{v} = \frac{2\pi a^{\frac{3}{2}}}{\sqrt{GM_S}} . \quad (174)$$

Kepler obtained these laws empirically and it took until Newton's time, that they were derived.
So far we have assumed the sun to be stationary, but the actual rotation is taking place around the commen center of mass of the sun and the planets. Since $M_{\odot} \approx 750 \sum m_{planets}$, this effect is small, but this effect is currently used to detect planets around other stars.

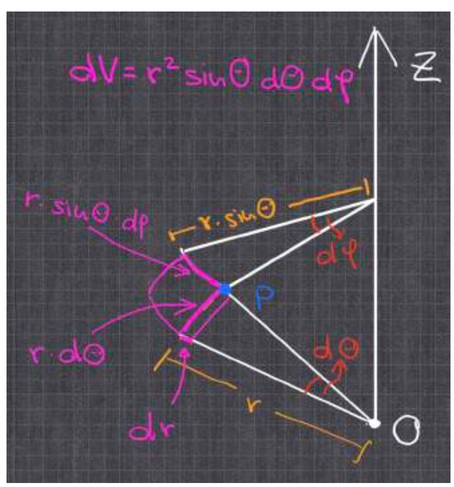
12.2 Spherical Mass Distribution

We were using several times the statement:
The gravitational attraction of a spherical mass distribution of mass M is identical to the gravitational attraction of apoint mass with value M, if the test mass is outside the spherical mass distribution.
Now we are going to prove that!
Remember the spherical coordinates

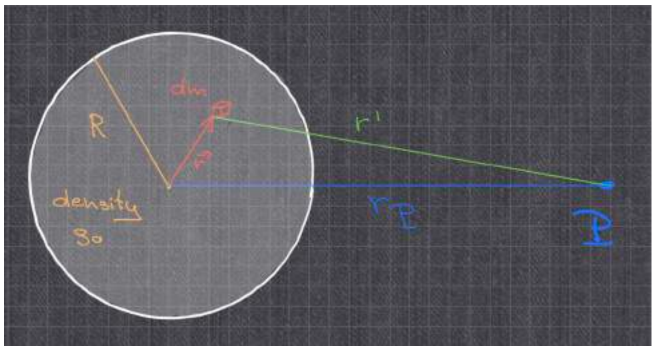


Yielding the volume element

$$dV = r^2 \sin\theta dr d\theta d\phi \tag{175}$$



Let us calculate the gravitational potential of a spherical mass distribution with radius R, total mass M and density ρ_0 at a point P outside the mass distribution.



The distance of point P and the centre of the mass distribution is denoted by r_P. We derive the gravitational potential by adding up the gravitational potentials steming from small mass elements dm at the position $\vec{r}$.

$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\sin\theta\sin\phi \\ r\sin\theta\cos\phi \\ r\cos\theta \end{pmatrix} . \tag{176}$$

The distance of these small mass elements at position $\vec{r}$ from the point P is denoted by r'. We get

$$r'^2 = x^2 + y^2 + (r_P - z)^2 \tag{177}$$

$$= r^2 - 2r \cdot r_P \cos\theta + r_P^2 . \tag{178}$$

Thus we get for the potential [6]

$$U = -\int\int\int \frac{Gdm}{r'} \tag{179}$$

$$= -G\rho_0 \int_0^R dr \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{r^2\sin\theta}{\sqrt{r^2 - 2rr_P\cos\theta + r_P^2}} \tag{180}$$

[6] We differentiate here between potential U and potential energy E_{pot}. The potential energy of a mass m at postion P is piven as $E_{pot} = U(\vec{r} = P) \times m$. In other words: the potential is the potential energy per unit mass. This concept will be used a lot in electro-statics and electro-dynamics, where mass is replaced by charge.

$$= \quad -2\pi G\rho_0 \int\limits_0^R drr^2 \int\limits_0^\pi d\theta \frac{\sin\theta}{\sqrt{r^2 - 2rr_P\cos\theta + r_P^2}} \tag{181}$$

$$= \quad -2\pi G\rho_0 \int\limits_0^R r^2 dr \left[\sqrt{r^2 - 2rr_P\cos\theta + r_P^2}\left(\frac{-1}{r\cdot r_P}\right)\right]_0^\pi \tag{182}$$

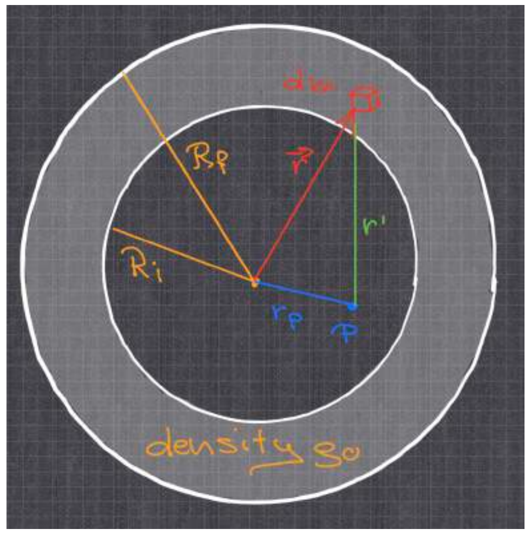
$$= \quad 2\pi G\rho_0 \int\limits_0^R \frac{r}{r_P} dr \left[\sqrt{(r_P + r)^2} - \sqrt{(r_P - r)^2}\right] \tag{183}$$

$$= \quad 2\pi G\rho_0 \int\limits_0^R \frac{r}{r_P} dr \left[r_P + r - (r_P - r)\right] \tag{184}$$

$$= \quad -4\pi G\rho_0 \int\limits_0^R \frac{r^2}{r_P} dr = -\frac{4}{3}\pi R^3 \rho_0 G \frac{1}{r_P} = -\frac{GM}{r_P}\,. \tag{185}$$

This is the potential of a pointlike mass M at the point P!
Be aware that we were using the fact that $r_P > r$ in the above derivation (Eq. (183) to Eq. (184)).

Next we want to prove: *for the gravitational attraction within a spherical symmetric mass distribution only the inner mass contributes*
Now we consider a point P inside a speherical mass distribution that is extending from radius R_i to radius R.



Thus we get for the potential - the derivation is almost identical with the essential difference that we have now $r_P < R$ (Eq. (187) to Eq. (188)):

$$U = -G\rho_0 \int\limits_{R_i}^{R_f} dr \int\limits_0^{\pi} d\theta \int\limits_0^{2\pi} d\phi \frac{r^2 \sin\theta}{\sqrt{r^2 - 2rr_P\cos\theta + r_P^2}} \tag{186}$$

$$= 2\pi G\rho_0 \int\limits_{R_i}^{R_f} \frac{r}{r_P} dr \left[\sqrt{(r_P + r)^2} - \sqrt{(r - r_P)^2}\right] \tag{187}$$

$$= 2\pi G\rho_0 \int\limits_{R_i}^{R_f} \frac{r}{r_P} dr \left[r_P + r - (r - r_P)\right] \tag{188}$$

$$= 4\pi G\rho_0 \int\limits_{R_i}^{R_f} r dr = 2\pi G\rho_0 \left(R_f^2 - R_i^2\right) . \tag{189}$$

Thus the potential inside the radius R_i is constant and does not depend on the position of P - deriving the gravitational force acting on the point P we will get zero!

12.3 Apparent Weight and the Earth's Rotation

Gravitational attraction on the equator

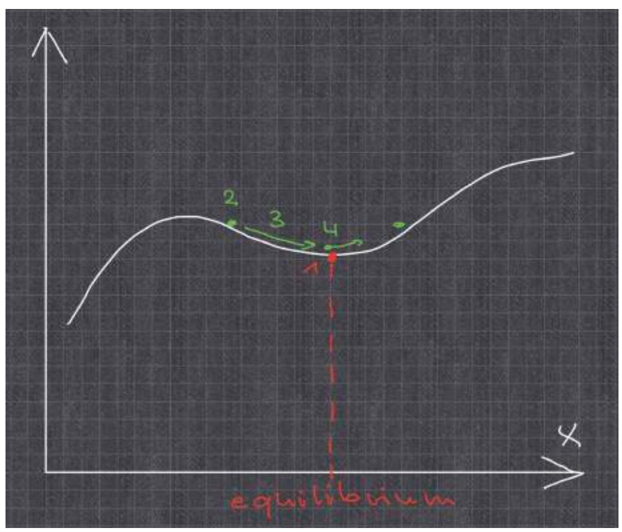
$$
\begin{aligned}
g &= g_0 - \frac{v^2}{R_E} && (190)\\
&= 9.82\frac{m}{s^2} - 0.0339\frac{m}{s^2} = 9.79\frac{m}{s^2}\,. && (191)
\end{aligned}
$$

13 Lecture 12: Periodic Motion 1

Textbook pages 437 - 471

E.g. swinging of a pendulum, sound vibrations, vibration of a quartz crystal, ...
Basic idea:

1. Start from an eliquibrium position of the oscillating object.
2. Move the object a little away from this position.
3. Let it move backwards.
4. A the equilibrium point there is still some kinetic energy, thus the object overshoots and moves to the other side.
5. ...



13.1 Describing Oscillations

Claim: A periodic oscillation looks like

$$x = A \sin(\omega t) \; . \tag{192}$$

a) A is the **amplitude** of the oscillation, i.e. the maximum displacement from the euilibrium position.

b) The **period** T of the oscillation is the time one cycle takes. The **frequency** is the number of cycles per unit time.

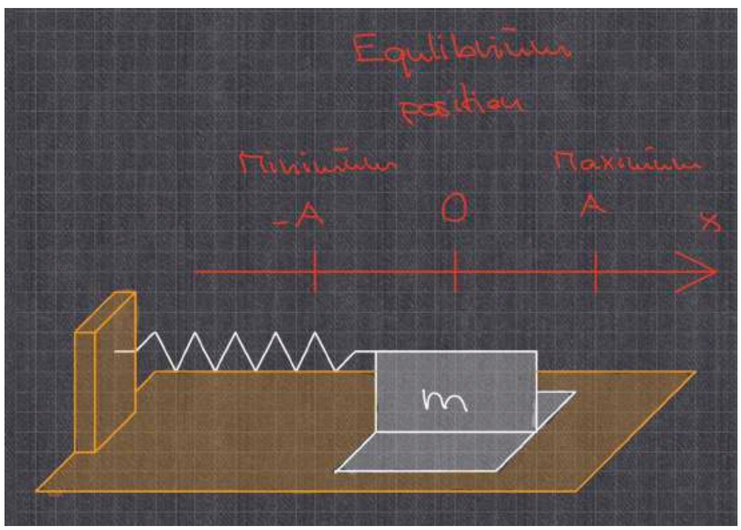
$$f = \frac{1}{T}, \qquad 1\text{hertz} = 1Hz = 1s^{-1}. \tag{193}$$

The **angular frequency** ω is

$$\omega = 2\pi f = \frac{2\pi}{T}. \tag{194}$$

A is the amplitude of the oscillation, i.e. the maximum displacement from the euilibrium position.

Example: A body of mass m resting on frictionless plane, that is attached to a spring



13.2 Simple Harmonic Motion

For a spring we have

$$F_x = -kx. \tag{195}$$

Thus we get for Newton's 2nd law:

$$ma_x = -kx \tag{196}$$

$$a_x = \frac{d^2x}{dt^2} = -\frac{k}{m}x \tag{197}$$

$$\Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0. \tag{198}$$

This defines a **simple harmonic oscillation (SHO)**. A body undergoing a SHO is called a **harmonic oscillator**. The most general solution of Eq.(198) is

$$x(t) = A\sin(\omega t + \phi_0) \ , \tag{199}$$

$$v_x = \frac{dx}{dt} = \omega A\cos(\omega t + \phi_0) \ , \tag{200}$$

$$a_x = \frac{d^2x}{dt^2} = -\omega^2 A\sin(\omega t + \phi_0) \ . \tag{201}$$

Inserting in Eq.(198) we get

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \tag{202}$$

$$-\omega^2 A\sin(\omega t + \phi_0) + \frac{k}{m}A\sin(\omega t + \phi_0) = 0 \tag{203}$$

$$\left(\frac{k}{m} - \omega^2\right) A\sin(\omega t + \phi_0) = 0 \tag{204}$$

$$\Leftrightarrow \omega = \sqrt{\frac{k}{m}} \ . \tag{205}$$

Thus we have have for the most general solution of the SHO:

$$x(t) = A\sin\left(\sqrt{\frac{k}{m}}t + \phi_0\right) \ , \tag{206}$$

$$x(0) = A\sin(\phi_0) \ , \tag{207}$$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \ , \tag{208}$$

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}} \ . \tag{209}$$

A and ϕ_0 can also be expressed in terms of the initial values of x and v:

$$x_0 = A\sin(\phi_0) \ , \tag{210}$$

$$v_0 = \omega A\cos(\phi_0) \ . \tag{211}$$

1.

$$\frac{x_0\omega}{v_0} = \tan\phi_0 \Rightarrow \phi_0 = \arctan\left(\frac{x_0\omega}{v_0}\right) \ . \tag{212}$$

2.

$$\sin^2\phi_0 + \cos^2\phi_0 = 1 \quad\Rightarrow\quad \frac{x_0^2}{A^2} + \frac{v_0^2}{\omega^2 A^2} = 1 \tag{213}$$

$$\Rightarrow\quad A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}\,. \tag{214}$$

13.3 Energy in Simple Harmonic Motion

For the sum of kinetic end potential energy we get

$$E = E_{kin} + E_{pot} \tag{215}$$

$$= \frac{m}{2}v^2 + \frac{k}{2}x^2 \tag{216}$$

$$= \frac{m\omega^2}{2}A^2\cos^2(...) + \frac{k}{2}A^2\sin^2(...) \tag{217}$$

$$= \frac{k}{2}A^2\left[\sin^2(...) + \cos^2(...)\right] \tag{218}$$

$$= \frac{k}{2}A^2 = \frac{k}{2}\left(x_0^2 + \frac{v_0^2}{\omega^2}\right) = \frac{k}{2}x_0^2 + \frac{m}{2}v_0^2\,. \tag{219}$$

The equation

$$kA^2 = mv^2 + kx^2 \tag{220}$$

can be used to derive a relation between v and x:

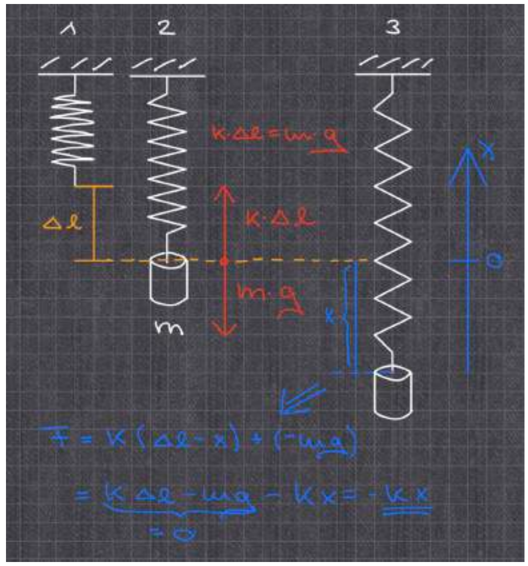
$$v = \sqrt{\frac{k}{m}\left(A^2 - x^2\right)}\,. \tag{221}$$

For the maximum speed we get

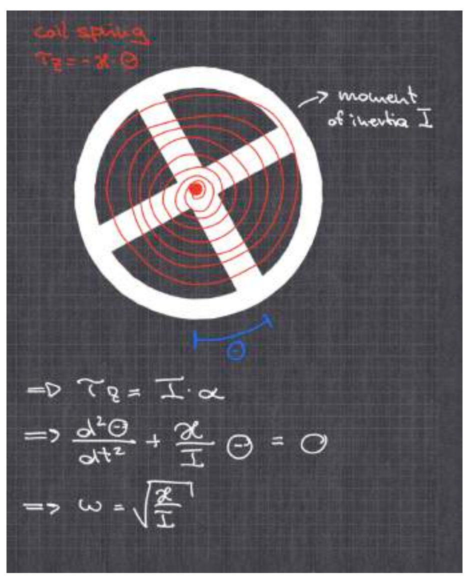
$$v_{max} = \sqrt{\frac{k}{m}}A = \omega A\,. \tag{222}$$

13.4 Applications of Simple Harmonic Motion

1. Vertical SHM



2. Angular SHM



3. Vibrations of Molecules

   The attraction/repulsion between two atoms in a molecues can be described by the van der Waals interaction, its potential is given by

$$U(r) = U_0 \left[\left(\frac{R_0}{r}\right)^{12} - 2\left(\frac{R_0}{r}\right)^{6} \right]. \tag{223}$$

   The left term describes repulsion, which dominates for small disctances and the right term results in attraction.
   The resulting force is given by

$$F_r = -\frac{dU(r)}{dr} = 12\frac{U_0}{R_0} \left[\left(\frac{R_0}{r}\right)^{13} - \left(\frac{R_0}{r}\right)^{7} \right]. \tag{224}$$

   The force vanishes at $r = R_0$ - this is an equlilibrium position. So, we investigate what is happening at small deviations from R_0: $r = R_0 + x$, for small values of x

$$\begin{aligned} F_r &= 12\frac{U_0}{R_0} \left[\left(\frac{R_0}{R_0 + x}\right)^{13} - \left(\frac{R_0}{R_0 + x}\right)^{7} \right] && (225) \\ &= 12\frac{U_0}{R_0} \left[\left(\frac{1}{1 + \frac{x}{R_0}}\right)^{13} - \left(\frac{1}{1 + \frac{x}{R_0}}\right)^{7} \right] && (226) \\ &\approx 12\frac{U_0}{R_0} \left[\left(1 - 13\frac{x}{R_0}\right) - \left(1 - 7\frac{x}{R_0}\right) \right] && (227) \\ &= -72\frac{U_0}{R_0^2} x\,. && (228) \end{aligned}$$

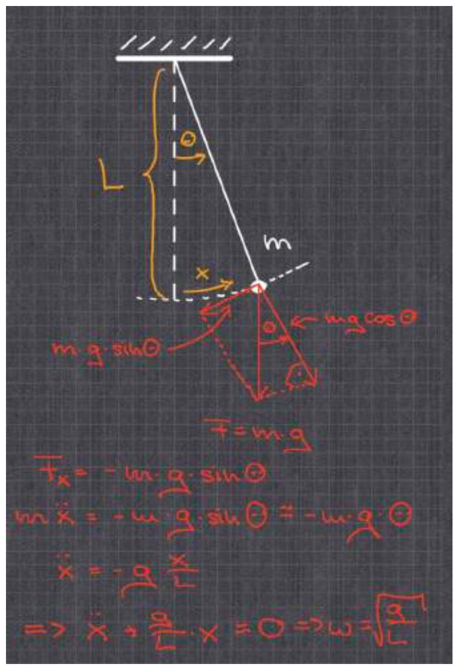
This is again Hooke's law! Thus we will get SHO!

14 Lecture 13: Periodic Motion 2

Textbook pages 437 - 471

14.1 The Simple Pendulum

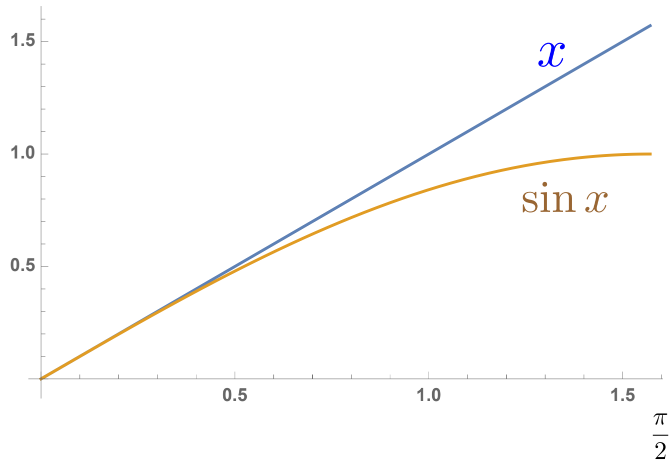
Point mass, suspended by a massless, unstretchable string.
E.g. weight on a cran cable, person on a swing, leg while walking,...



Here we have used the Taylor expansion of the sin. You will learn that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \tag{229}$$

For small values of x (in radians!) already x gives a very good approximation of the sin.



One can show that the osciallation period can be written generally as

$$T = 2\pi\sqrt{\frac{L}{g}}\left[1 + \frac{1}{4}\sin^2\frac{\Theta}{2} + \frac{9}{64}\sin^4\frac{\Theta}{2} + ...\right], \tag{230}$$

with the maximum angular displacement Θ.

14.2 The Physical Pendulum

Extended object, center of mass is in the distance d of the pivot:

$$\omega^2 = \frac{mgd}{I}. \tag{231}$$

For $I = md^2$ this reduces to the well-known formula $\omega^2 = g/d$.
Walking of animals can be described by harmonic oscillations of a physical pendulum = leg!

14.3 Damped Oscillations

Simplest case: friction proprtional to the force.

$$ma_x = \sum F_x = -kx - bv_x \tag{232}$$

$$\ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} = 0. \tag{233}$$

The solution of this differential equation is given by

$$
\begin{aligned}
x &= Ae^{-\gamma t}\sin(\tilde{\omega}t)\,, && (234)\\
\dot{x} &= Ae^{-\gamma t}\left[-\gamma\sin(\tilde{\omega}t)+\tilde{\omega}\cos(\tilde{\omega}t)\right]\,, && (235)\\
\ddot{x} &= Ae^{-\gamma t}\left[\gamma^2\sin(\tilde{\omega}t)-\gamma\tilde{\omega}\cos(\tilde{\omega}t)\right]+Ae^{-\gamma t}\left[-\gamma\tilde{\omega}\cos(\tilde{\omega}t)-\tilde{\omega}^2\sin(\tilde{\omega}t)\right]\\
&= Ae^{-\gamma t}\left[(\gamma^2-\tilde{\omega}^2)\sin(\tilde{\omega}t)-2\gamma\tilde{\omega}\cos(\tilde{\omega}t)\right]\,. && (236)\\
\Rightarrow 0 &= \ddot{x}+\frac{k}{m}x+\frac{b}{m}\dot{x} && (237)\\
&= Ae^{-\gamma t}\left[\left(\gamma^2-\tilde{\omega}^2+\frac{k}{m}-\gamma\frac{b}{m}\right)\sin(\tilde{\omega}t)+\left(\tilde{\omega}\frac{b}{m}-2\gamma\tilde{\omega}\right)\cos(\tilde{\omega}t)\right]\,. && (238)
\end{aligned}
$$

For this to be always zero we have the two requirements

$$
\begin{aligned}
\tilde{\omega}\left(\frac{b}{m}-2\gamma\right) &= 0 \Leftrightarrow \gamma=\frac{b}{2m}\,, && (239)\\
\left(\gamma^2-\tilde{\omega}^2+\frac{k}{m}-\gamma\frac{b}{m}\right) &= 0 \Leftrightarrow \tilde{\omega}^2=\frac{k}{m}-\frac{b^2}{4m^2}\,. && (240)
\end{aligned}
$$

Thus the most general solution reads

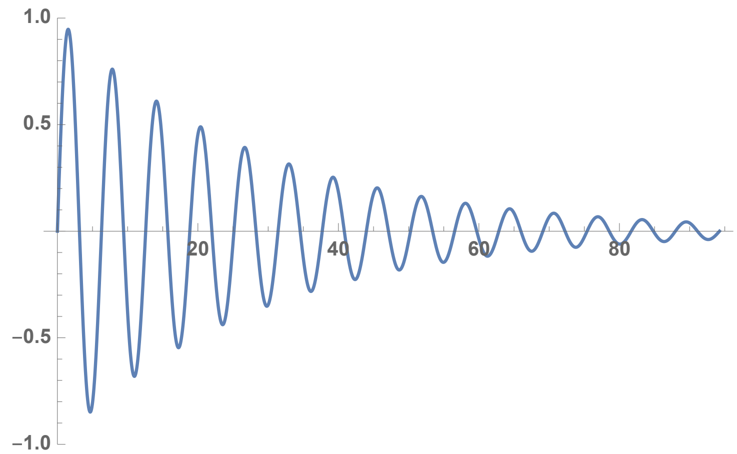
$$
x = Ae^{-\frac{b}{2m}t}\sin\left(\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}\,t+\phi_0\right)\,. \qquad (241)
$$

Remarks: in comparison to the undamped case:

1. The Amplitude decreases with time: $Ae^{-\frac{b}{2m}t}$.

2. The oscillation frequency is smaller than in the free case.

$$
\frac{k}{m}-\frac{b^2}{4m^2} < \frac{k}{m}\,. \qquad (242)
$$



- if $b^2 < 4km$ **underdamping**,
- the oscillation frequency becomes zero if $b^2 = 4km$, which is called **critical damping**,
- if $b^2 > 4km$ we have **overdamping**, in this case the general soultion will look like

$$x = C_1 e^{-\gamma_1 t} + C_2 e^{\gamma_2 t} . \tag{243}$$

The energy of the damped system reads

$$E = \frac{m}{2} v_x^2 + \frac{k}{2} x^2 . \tag{244}$$

The time derivative of the energy reads

$$\begin{aligned} \frac{dE}{dt} &= m v_x \dot{v}_x + k x \dot{x} \\ &= v_x \left(m a_x + k x \right) \\ &= v_x \left(-b v_x \right)) = -b v_x^2 . \end{aligned} \tag{245}$$

Thus the energy is becoming less.

14.4 Forced Oscillations and Resonance

- System with oscillation freuqency $\tilde{\omega}$

- Driving force with frequency ω_d

$$F_d = F_{max} \sin \omega_d t \,. \tag{246}$$

A detailed analysis for the amplitude gives

$$A = \frac{F_{max}}{(k - m\omega_d^2)^2 + b^2\omega_d^2} \,, \tag{247}$$

with its maximum at $\omega_d = \tilde{\omega} = \sqrt{\frac{k}{m}}$ (**resonance**) .

15 Lecture 15: Revision, Examples, Outlook

15.1 Rotation of Rigid Bodies

Angular movement vs. linear movement

$$x(t) = \frac{1}{2}at^2 + v(t_0)t + x_0\,, \tag{248}$$

$$\theta(t) = \frac{1}{2}\alpha t^2 + \omega(t_0)t + \theta_0\,. \tag{249}$$

Roational energy

$$E_{kin} = \sum_i \frac{1}{2}m_i v_i^2 = \frac{1}{2}\omega^2 \sum_i m_i r_i^2 =: \frac{1}{2}\omega^2 I \tag{250}$$

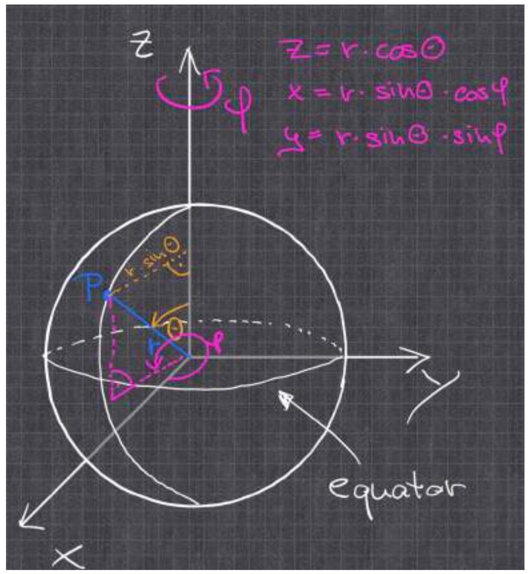
- Moment of inertia

$$I = \sum_i r_i^2 m_i \to \int r^2 \rho dV \tag{251}$$

- Parallel axis theorem

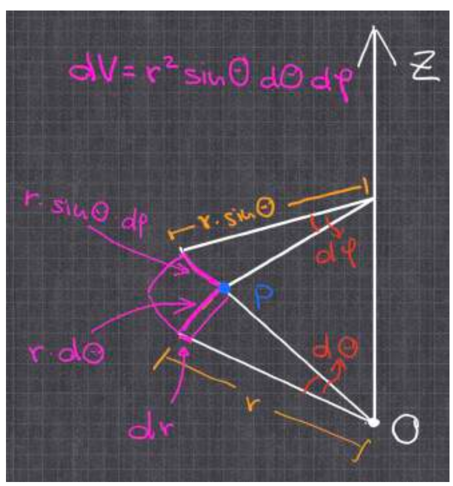
$$I_P = I_{C.M.} + MR^2 \tag{252}$$

- Spherical coordinates
  Can either be done mathematically with Jacobian determinant or graphically



Yielding the volume element

$$dV = r^2 \sin\theta dr d\theta d\phi \tag{253}$$



- Multi-dimensional integrals

$$I = \int_0^1 d\alpha \int_0^2 d\beta \int_0^3 d\gamma \int_0^4 d\delta \cdot \alpha^2 \beta^3 (\gamma - \delta\alpha) = 0 \tag{254}$$

Torque (e.g. for moving cylinder...)

$$\vec{\tau} = \vec{r} \times \vec{F} \tag{255}$$

Rotation around a moving axis = Translation plus Rotation

15.2 Dynamics of Rotational Motion

Conserved quantities: rotational energy E_{rot} and angular momentum $\vec{L}$

$$E_{rot} = \frac{1}{2} I \omega^2 \tag{256}$$

$$\vec{\mathrm{L}} = \vec{r} \times \vec{p} \tag{257}$$

Kepler laws, Gyroscopes,...

15.3 Equilibrium and Elasticity

Equilibrium condition

$$\sum \vec{F} = \vec{0} = \sum \vec{\tau} \tag{258}$$

Slipping ladder
Stress, Strain and elastic modulus: tensile, bulk and shear

15.4 Fluid Mechanics

Pascal's law (incompressible fluid)

$$p = p_0 + \rho g h \tag{259}$$

Hydraulic press
Bouyancy

Continuity equation: conservation of mass - will be very impartant in Quantum mechanics and Quantum Field Theory

$$\rho_i A_i v_i = const. \tag{260}$$

Bernoulli equation

$$p + \rho g y + \frac{1}{2}\rho v^2 = const \tag{261}$$

Viscosity and turbulence - Navier Stokes equations are still not solved!

15.5 Gravitation

Newtons law - gravitation is very weak

$$F = mg \quad vs \quad F = \frac{Gm_1 m_2}{r^2} \tag{262}$$

$$E = mgh \quad vs \quad E = -Gm_1 m_2 \left(\frac{1}{r_2} - \frac{1}{r_1}\right) \tag{263}$$

Escape velocity - Schwarzschildradius - Black holes
Motion of satellites (dark matter) - motion of planets (Kepler laws)
Spherical mass distribution: only the mass inside counts!!!!

15.6 Periodic Motion

One of the most fundamental approximations in physics. All boils down to equations like

$$\ddot{x} + \frac{k}{m}x + \frac{b}{m}\dot{x} \;\; = \;\; 0\,. \tag{264}$$

Remember $\sin^2 x + \cos^2 x = 1$.

16 Acknowledgements

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References

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