Cosmology 1
CPT

Michaelmas 2018

Per Aspera ad Andromeda

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IPPP Durham

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Contents

1 Introduction

Dear students,

I am very much looking forward to teach this course. My specialisation is Heavy Flavour Physics - so obvioulsy not cosmology. But one of the main motivations for my research field is to try to find out the origin of the matter-antimatter asymmetry in the Universe. According to the Sakharov criteria CP violation is a necessary ingredient of the fundamental laws of nature in order to create a baryon asymmetry when starting from symmetric initial conditions. The study of CP violation in heavy hadron decays is a main part of my research.
Another connection of my research field to cosmology is the nature of dark matter. Many physicists expect dark matter to consist of new elementary particles, that will also couple to some extent to the known particles of the Standard Model of Particle Physics. Decays of heavy hardons are also used to find traces of tiny couplings between the SM sector and the dark sector.
After discusing with cosmologists from the ICC and with some of my colleagues at the IPPP and the Maths department, the idea is to present in *Cosmology 1* a basic introduction into the topic - the minimum every particle physicist should know. Therefore I will closely follow the book *An Introduction to Modern Cosmology* of Andrew Liddle. You already had the *General Theory of Relativity Lecture* taught by my colleagues from the Maths department and you will have the more advanced *Cosmology 2* lecture.
In case you want to contact me outside the lectures: my email address is alexander.lenz@durham.ac.uk and my room number is OC121 in the ground floor of the Ogden building (the building where also OC218 is located).
I will provide my lecture notes after the lecture on DUO - please let me know about any misprints you find in the notes.
Again, I am looking forward to do this exciting expidition into modern Cosmology with you in the next four weeks.

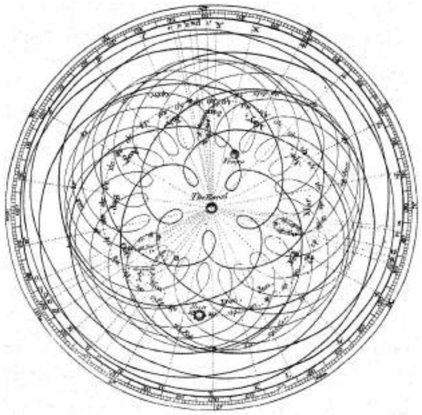
Alexander Lenz

2 Lecture 1: Overview

Mo 12.11.: Chapter 1, 2

2.1 The Cosmological Principle

A very brief history of the centre of the Universe:



- Ancient Greeks like Ptolemy: Earth is in the centre (epicycles).
- 1500s Copernicus: Sun is in the centre.
- Newton: empirical science ⇒ Theory of Gravitation.
- 1700s William and Caroline Herschel: Solar system in the centre of the Milky Way.
- Early 1900s Shapley: sun away from the centre of the Milky Way; but Milky way in the centre of the Universe.
- 1952 Baade: Milky way is just one among billions of other galaxies.

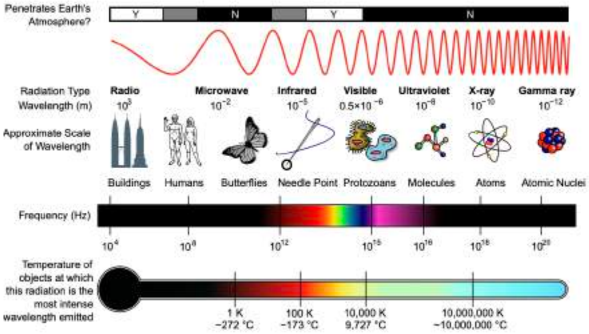
Modern view: On very large scales (e.g. a million galaxies and above) the Universe is the same everywhere.
This **cosmological principle** is the main pillar of the **Big Bang** theory.
It is of course violated on smaller scales!

2.2 Observation

2.2.1 Observational Channels

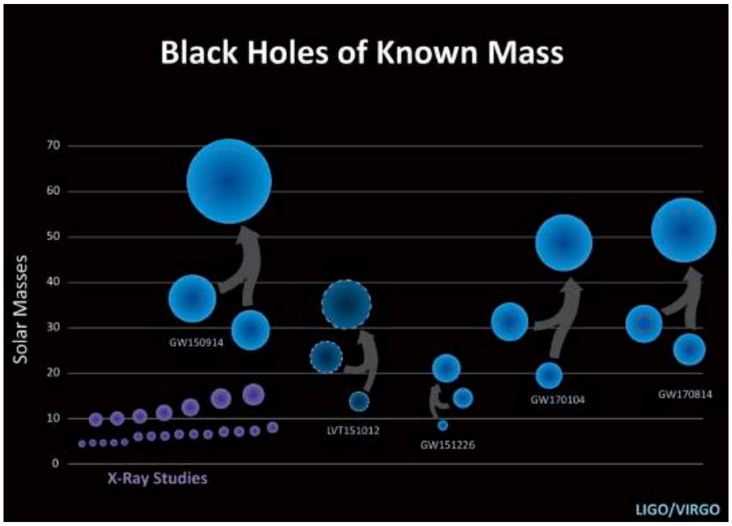
Astrophysical observations are done via:

- Since ancient times: visible light.
- Since the 19th century: full electro-magnetic spectrum
  radio waves, microwaves, IR, visible light, UV, x-ray, gamma.

  Both ground based and satellite.

- Recently also:
  - From 1909 onwards (1912 Hess): Cosmic rays.
  - 1987: Neutrinos SN1987A (Kamiokande II detected 12 antineutrinos; IMB, 8 antineutrinos; and Baksan, 5 antineutrinos).
  - 2016: Gravitational waves - LIGO 5 events so far:

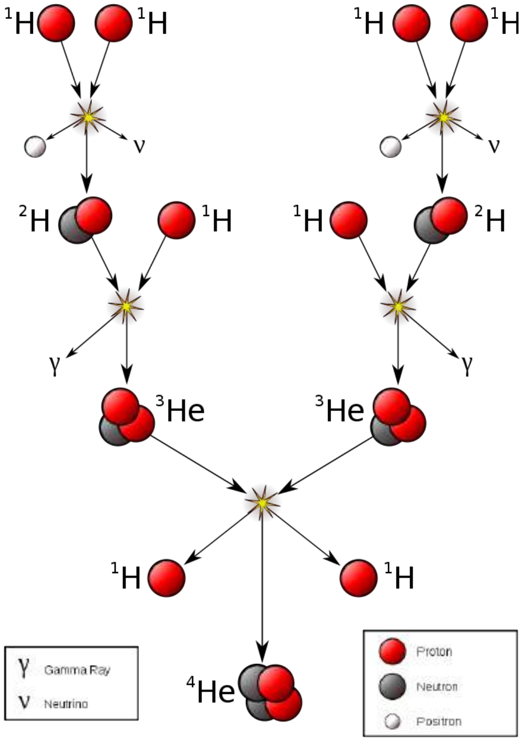


https://www.ligo.caltech.edu/page/detection-companion-papers

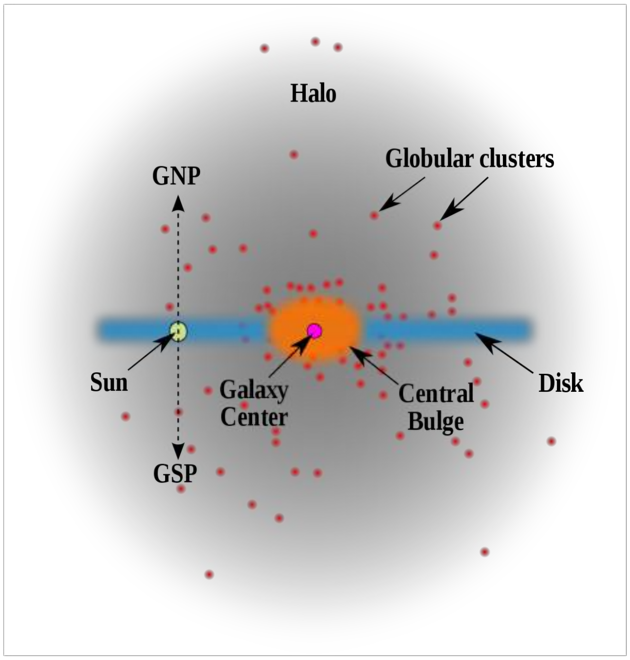
2.2.2 Observations in the Visible Spectrum

What did we learn from observations in the visible spectrum?

Stars: create energy by nuclear fusion, sun is a typical star, solar mass $M_\odot = 2 \cdot 10^{30}$ kg. The closest star is Proxima Centauri with a distance of 4.22 light years (1 light year = $9.46 \cdot 10^{15}$ m; 1 parsec (pc) = 3.26 light-years).



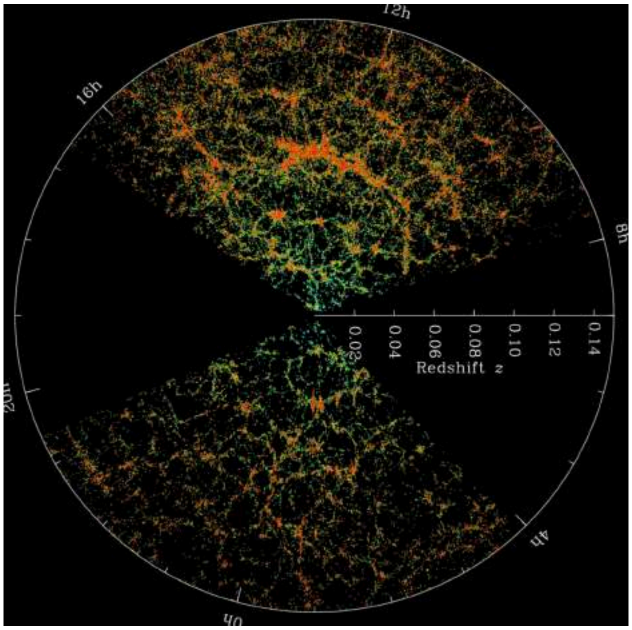
Galaxies: The Milky Way (MW) contains about 10^{11} stars with masses ranging from $1/10 M_\odot$ to almost $100 M_\odot$. The disc of the MW has a radius of 12.5 kpc and a thickness of 0.3 kpc. The sun is about 8kpc away from the centre and it rotates in about 200 million years around the centre. The bulge is surrounded by globular clusters: about a million stars, symetrically around the bulge in distances of 5pc to 30 kpc.



The study of galaxies gives hints for the existence of dark matter

The local group: Large Magelanic cloud (50 kpc), Andromeda (770 kpc).
Typical galaxy group: $V = \mathcal{O}(Mpc^3)$.

Galaxy clusters: $\mathcal{O}(100Mpc)$ - *2dF galaxy red shift survey*, *Sloan Digital Sky Survey*:

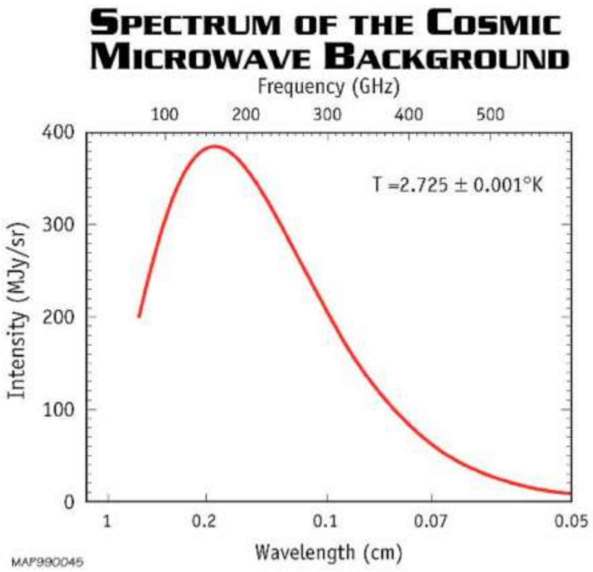


Virgo Cluster	Centre $16 Mpc$ away, about 1300 galaxies
Coma Cluster	Centre $100 Mpc$ away, more than 1000 identified galaxies, maybe 10000
−	most galaxies do not belong to a cluster - field galaxies
Supercluster	Groups of clusters, e.g. Virgo Supercluster - local group is a member, or Coma Supercluster
Voids	50 MpC
Large scale smoothness	Only recently confirmed by measurement.

2.2.3 Observations in the Non-visble Spectrum

What did we learn for other wave lengths?

Microwave:	Cosmic Microwave background = afterglow of the Big Bang; 1965 accidentally discovered by Penzias and Wilson. Black body spectrum with a temperature $T = 2.725 \pm 0.001 K$.



	Very precisely measured by e.g. COBE, WMAP, PLANCK - tiny anisotropies ($\mathcal{O}(10^{-5})$) give clues about the very early Universe.
IR	Look through dust (e.g. galactic plane) and at old galaxies (red-shift)
X-ray	Hot gas in galaxie clusters
Radio	21cm line of Hydrogen - distribution of hydrogen in distant parts of the Universer

All in all our Universe seems to be **homogenous** and **isotropic**.

2.2.4 Red-shift

The light of distant galaxies is red-shifted. The red shift z is defined as

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \,. \tag{1}$$

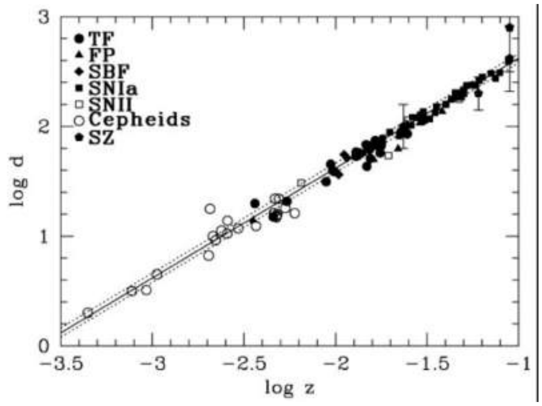
The redshift is assumed to stem from the Doppler effect and one gets the following relation to the velocity of the galaxy:

$$z = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} - 1 \approx \frac{v}{c}\,, \quad \text{if} \, \frac{v}{c} \ll 1 \,. \tag{2}$$

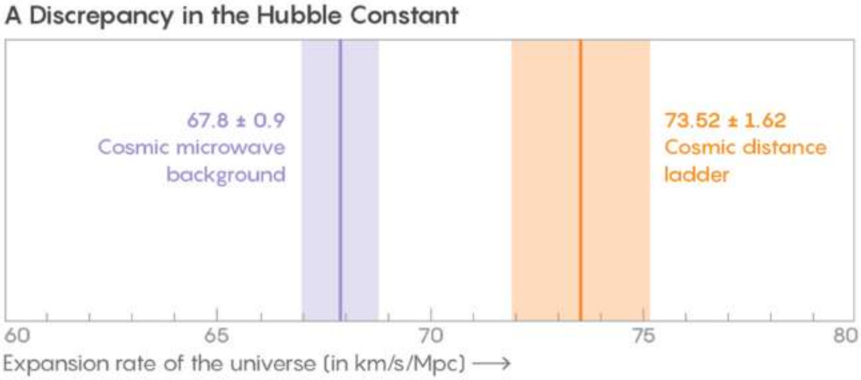
Observation gives a linear relation velocity of the galaxy and distance of the galaxy

$$\vec{v} = H_0 \vec{r}, \tag{3}$$

with the Hubble constant H_0.



For nearby galaxies the Hubble law holds of course worse - the farer a galaxy away, the better the law holds.
The red-shift provides very strong evidence for the Big Bang theory!
Most recent determinations show some minor discrepancies in the determination of the exact value of the Hubble constant.



2.2.5 Particles in the Universe

The energy of a particle is given as

$$E = \sqrt{m^2c^4 + p^2c^2} = mc^2\sqrt{1 + \frac{p^2}{m^2c^2}} \approx mc^2 + \frac{p^2}{2m}\,, \qquad (4)$$

where the approximation holds for non-relativistic particles.
We find **baryons** in the Universe:

$$p = uud \qquad 938.3\,MeV \qquad (5)$$

$$n = ddu \qquad 939.6\,MeV \qquad (6)$$

Cosmologists include also electrons to baryons! ($m_e = 511\,keV$).
Radiation consists of **photons** γ propagating with c. The energy of a photon with frequency f is given as

$$E = hf\,. \qquad (7)$$

Neutrinos are very weekly interacting, very light, i.e. they are treated as relativistic particles - therfore cosmologists call them sometimes also radiation! Despite its smallness, the neutrino mass might have an observable

effect cosmological effect.
Dark matter will be discussed later.

In **thermal equilibrium** particles are interacting frequently and there is a balance of forward and backward reactions. The overall distribution of particle number and energy remains fixed.
There are different distributions for **bosons** and **fermions**. For photons we get the **Planck** or **black body** spectrum. Each of the two photon polarisations has an **occupation number** per mode $\mathcal{N}$:

$$\mathcal{N} = \frac{1}{\exp\left[\frac{hf}{k_b T}\right] - 1}\,, \tag{8}$$

with the **Boltzmann constant** $k_b = 1.281 \cdot 10^{-23} J/K = 8.619 \cdot 10^{-5} eV/K$.
The Planck function states that photons with energies smaller than $k_B T$ can easliy be created, but photon woth larger energies are very rare.
The energy density ϵ (energy per unit volume) in a frequency interval df around f is given by

$$\epsilon(f)df = \frac{8\pi h}{c^3} \frac{f^3 df}{\exp\left[\frac{hf}{k_b T}\right] - 1}\,. \tag{9}$$

The peak of this distribution is at $E_{Peak} = 2.8 k_B T$, the mean energy of a photon is given by $E_{Peak} = 3 k_B T$.
For the history of the Universe it will be important how typical atomic or nuclear binding energies compare to these energies. The overall energy is given by

$$\epsilon_{rad} = \frac{8\pi k_B^4}{h^3 c^3} T^4 \int_0^\infty \frac{y^3 dy}{e^y - 1} = \frac{8\pi^5 k_B^4}{15 h^3 c^3} T^4 = \alpha T^4\,. \tag{10}$$

3 Lecture 2: Modelling the Universe I

We 14.11.: Chapter 3,4

3.1 Newtonian Gravity

Derive the evolution of the Universe without general relativity - fill later the loopholes.

3.1.1 Introduction

Newton:

$$F_G = \frac{GMm}{r^2} \,. \tag{11}$$

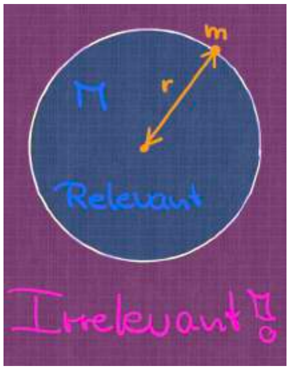
Any force creates an acceleration according to $F = ma$, thus the gravitational acceleration of a body with mass m is independent of this mass!
The exerted force can also be described by a potential

$$V(r) = -\frac{GMm}{r} \,. \tag{12}$$

Gravity always attracts (negative potential) and the force is in the direction of the steepest decrease of the potential.

$$\vec{F} = -\nabla V(r) = -\frac{GMm}{r^2}\frac{\vec{r}}{r} \,. \tag{13}$$

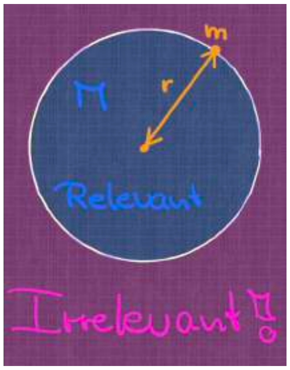
Newton: in a spherical symmetric mass distribution a particle feels no effects of all the mass at a greater radius than its distance from the origin. All the effect of the masses from smaller radii is equivalent to a case where all this mass is concentrated at the origin.



3.1.2 Friedmann Equation

Most important equation in cosmology!
Standard task in cosmology = solving this equation with different assumptions concerning the matter content.

- Consider uniform expanding medium with mass density $\rho = Mass/Volume$.
- Cosmological Principle: Any point can be considered to be the centre
- Consider a particle of mass m at a distance r from the centre



- this feels only the effect of the mass M

$$M = \frac{4}{3}\pi r^3 \rho \,. \tag{14}$$

Thus is feels the force F

$$F = \frac{GMm}{r^2} = \frac{4}{3}G\pi r m \rho \tag{15}$$

and it has the potential energy

$$V = \frac{GMm}{r} = -\frac{4}{3}G\pi r^2 m \rho \,. \tag{16}$$

- The particle has the kinetic energy

$$T = \frac{1}{2}m\dot{r}^2 \tag{17}$$

- We get for the conserved total energy

$$U = T + V = \frac{1}{2}m\dot{r}^2 - \frac{4}{3}G\pi r^2 m \rho \,. \tag{18}$$

- Change to **comoving coordinates** ($\vec{x}$) = coordinates that are carried along with the expansion, i.e. they are not changed by the expansion of the Universe. The real distance $\vec{r}$ can be written as

  $$\vec{r} = a(t)\vec{x}\,, \tag{19}$$

  where $a(t)$ is the scale factor of the Universe, that depends only on time.

  $$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4}{3}G\pi a^2x^2m\rho\,. \tag{20}$$

  $\dot{x} = 0$ by definition as objects are fixed in comoving coordinates - multiply by $2/(ma^2x^2)$.

  $$\begin{aligned} \frac{2U}{ma^2x^2} &= \frac{\dot{a}^2}{a^2} - \frac{8}{3}G\pi\rho && (21)\\ \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8}{3}G\pi\rho - \frac{kc^2}{a^2}\,, Friedmann && (22)\end{aligned}$$

  with $kc^2 = -2U/(mx^2)$. This is the standard form of the **Friedmann Equation**.

3.1.3 The Fluid Equation

In order to make use of the Friedmann equation we need to know $\rho = \rho(t)$.

- The 1st law of thermodynamics reads

  $$dE + pdV = TdS\,. \tag{23}$$

- We want to apply this to an expanding volume V of unit comoving radius a. The energy of the volume is given by

  $$E = mc^2 = \frac{4}{3}\pi a^3\rho c^2\,. \tag{24}$$

  Thus we get for the change of energy per time

  $$\frac{dE}{dt} = 4\pi a^2\dot{a}\rho c^2 + \frac{4}{3}\pi a^3\dot{\rho}c^2\,. \tag{25}$$

  The change of volume per time is given by

  $$\frac{dV}{dt} = 4\pi a^2\dot{a}\,. \tag{26}$$

- Having a reversible, adiabatic process ($dS = 0$), we can rewrite the first law of thermodynamics as

$$\begin{aligned} 0 &= \frac{dE}{dt} + p\frac{dV}{dt} && (27)\\ &= 4\pi a^2 \dot{a}\rho c^2 + \frac{4}{3}\pi a^3 \dot{\rho} c^2 + p4\pi a^2 \dot{a} && (28)\\ 0 &= \dot{a}\rho + \frac{1}{3}a\dot{\rho} + \frac{p}{c^2}\dot{a} && (29)\\ \Rightarrow \qquad & \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0\,.fluid && (30) \end{aligned}$$

  This is the **Fluid Equation**.

The change of density can origin from a dilution in the density, because the volume increases (1st term) or from a loss of energy because the pressure of the material has done work as the volume increased (2nd term), which has gone into the gravitational energy.
In order to solve the equations we still need an **equation of state**, i.e. $p = p(\rho)$ (*see later*).

3.1.4 The Acceleration Equation

Time derivative on Friedmann Equation - Eq. (22)

$$2\frac{\dot{a}}{a}\left(\frac{a\ddot{a} - \dot{a}^2}{a^2}\right) = \frac{8}{3}G\pi\dot{\rho} + 2\frac{kc^2\dot{a}}{a^3}\,. \qquad (31)$$

Substitute $\dot{\rho}$ from Fluid Equation - Eq.(30)

$$\begin{aligned} 2\frac{\dot{a}}{a}\left(\frac{a\ddot{a} - \dot{a}^2}{a^2}\right) &= -8G\pi\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) + 2\frac{kc^2\dot{a}}{a^3} && (32)\\ \left(\frac{a\ddot{a} - \dot{a}^2}{a^2}\right) &= -4G\pi\left(\rho + \frac{p}{c^2}\right) + \frac{kc^2}{a^2}\,. && (33) \end{aligned}$$

Replace kc^2/a^2 via Friedmann Equation - Eq. (22)

$$\begin{aligned} \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} &= -4G\pi\left(\rho + \frac{p}{c^2}\right) + \frac{8}{3}G\pi\rho - \left(\frac{\dot{a}}{a}\right)^2 && (34)\\ \frac{\ddot{a}}{a} &= -4G\pi\left(\frac{\rho}{3} + \frac{p}{c^2}\right)\,.acceleration && (35) \end{aligned}$$

This is the **Acceleration Equation**. Any pressure increases the gravitational potential and thus decelerates the expansion. This equation does not depend on k!

3.2 Geometry of the Universe

What does k in the Friedmann equation mean?
Newton: energy per particle
Einstein (GTR): curvature of space
What geometries do fulfill isotropy and homogenity?

3.2.1 Flat Geometry = Euclidean Geometry

- A straight line is the shortest connection between two points.
- Two parallel lines stay parallel.
- Angles in a traingle sum up to 180°.
- The circumference of a circle with radius r is given by $2\pi r$.

A Universe with such a geometry is called a **flat Universe** - such a Universe must be infinite, else there will be an edge?
A flat Universe corresponds to $k = 0$.

3.2.2 Spherical Geometry

Riemann = non-Euclidean
A sphere is isotropic and homogenous

- Straight lines are segments of great circles.
- Two parallel lines do not have to stay parallel.
- Angles in a traingle sum up to more than 180°.
- The circumference of a circle with radius r is less than $2\pi r$.

Such an Universe is finite, without having a border!
At very small scales this looks again Euclidean.
A spherical Universe corresponds to a positive value of k, it is also called a closed Universe.

3.2.3 Hyperbolic Geometry

- Angles in a traingle sum up to less than 180°.
- The circumference of a circle with radius r is more than $2\pi r$.

A hyperbolic Universe corresponds to a negative value of k, it is also called an open Universe.

3.2.4 Infinite and Observable Universes

Observable Universe = part of the Universe we can observe: $L = c \cdot t$, where t is the age of the Universe.

3.2.5 Where did the Big Bang happen?

Everywhere!

3.2.6 Three Values of K

Use natural units and define

$$\begin{aligned} \hat{a} &= \frac{a}{\sqrt{k}} \,, & (36) \\ \Rightarrow \left(\frac{\dot{\hat{a}}}{\hat{a}}\right)^2 &= \frac{8}{3} G\pi\rho \pm \frac{1}{\hat{a}^2} \,. & (37) \end{aligned}$$

Choosing the plus sign, corresponds to negative k, i.e. a closed universe; the minus sign to an open universe and neglecting the last term to a flat universe.

4 Lecture 3: Modelling the Universe II

Mo 19.11.: Chapter 5,6

Start with

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}G\pi\rho - \frac{k}{a^2}\,, \tag{38}$$

$$0 = \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right)\,. \tag{39}$$

4.1 Simple Cosmological Models

4.1.1 Hubble's Law

The velocity of recession $\vec{v} = d\vec{r}/dt$ points in the same direction as the vector $\vec{r}$. Thus we can write

$$\begin{aligned} \vec{v} &= \frac{d\vec{r}}{dt} = \left|\dot{\vec{r}}\right| \frac{\vec{r}}{|\vec{r}|} = \frac{\dot{a}}{a}\vec{r} \\ &= H\vec{r}, \end{aligned} \tag{40}$$

with the Hubble constant

$$H = \frac{\dot{a}}{a}\,. \tag{41}$$

Thus the Hubble constant depends on time, and the value measured today is denoted as H_0. The Friedmann equation can thus also be written as

$$H(t)^2 = \frac{8}{3}G\pi\rho - \frac{k}{a^2}\,. \tag{42}$$

H might be better called **Hubble parameter** - H is constant in space but not in time!
?If we look at a galaxy that is $1MLyrs$ *away, do we measure* $H(t_{now})$ *or* $H(t_{now} - 1Myrs)$*?*

4.1.2 Expansion and Redshift

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} \Rightarrow 1 + z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{a(t_{obs})}{a(t_{em})} \tag{43}$$

4.1.3 Equations of State

1. **Matter:** i.e. non-relativistc matter (sometimes also called dust) that excerts negligible pressure, i.e.

   $$p = 0$$

   Good approximation for the atoms in a cool universe (only rare interaction) or for a collection of galaxies.

2. **Radiation:** phtons move with c, which leads to a preasure force, the radiation pressure:

   $$p = \frac{\rho c^3}{3}$$

   Also other particles moving at high velocities (e.g. neutrinos) have this equation of state.

4.1.4 Solving the Equations for Matter with $k = 0$

The fluid equation gives

$$\begin{aligned} \dot{\rho} + 3\frac{\dot{a}}{a}\rho &= 0 & (44)\\ \Leftrightarrow \frac{1}{a^3}\frac{d}{dt}\left(\rho a^3\right) &= 0 & (45)\\ \Leftrightarrow \frac{d}{dt}\left(\rho a^3\right) &= 0 & (46)\\ \Leftrightarrow \rho &= \frac{\rho_0}{a^3}\,. & (47)\end{aligned}$$

Unsurprising: density falls off with the volume of the Universe
For $k = 0$ the Friedmann equations are scale inviariant, i.e. a and ca have the same Friedmann equations
$\Rightarrow$ choose $a = 1$ today, then ρ_0 is the density today.

Inserting in the Friedmann-equation we get:

$$\begin{aligned} \left(\frac{\dot{a}}{a}\right)^2 &= \frac{8}{3}G\pi\frac{\rho_0}{a^3} & (48)\\ \dot{a} &= \sqrt{\frac{8G\pi\rho_0}{3}}\frac{1}{\sqrt{a}} & (49)\end{aligned}$$

$$
\begin{aligned}
da\sqrt{a} &= \sqrt{\frac{8G\pi\rho_0}{3}}dt & (50)\\
\left[\frac{2}{3}a^{\frac{3}{2}}\right]_0^t &= \sqrt{\frac{8G\pi\rho_0}{3}}t & (51)\\
\frac{2}{3}a^{\frac{3}{2}}(t) &= \sqrt{\frac{8G\pi\rho_0}{3}}t & (52)\\
\left[\frac{3}{2}\sqrt{\frac{8G\pi\rho_0}{3}}t\right]^{\frac{2}{3}} &= a(t) & (53)\\
\Rightarrow \frac{a(t)}{a(t_0)} = a(t) &= \left(\frac{t}{t_0}\right)^{\frac{2}{3}} . & (54)
\end{aligned}
$$

Now we have also the time evolution of the density

$$
\rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2} , \qquad (55)
$$

and the Hubble parameter

$$
H = \frac{\dot{a}}{a} = \frac{2}{3}\frac{t^{-\frac{1}{3}}}{t_0^{\frac{2}{3}}}\frac{t_0^{\frac{2}{3}}}{t^{\frac{2}{3}}} = \frac{2}{3}\frac{1}{t} . \qquad (56)
$$

4.1.5 Solving the Equations for Radiation with $k = 0$

Radiations obeys $p = \rho c^2/3$ thus we get from the fluid equation

$$
\begin{aligned}
\dot{\rho} + 4\frac{\dot{a}}{a} &= 0 & (57)\\
\frac{1}{a^4}\frac{d}{dt}\left(\rho a^4\right) &= 0 & (58)\\
\rho &= \frac{\rho_0}{a^4} . & (59)
\end{aligned}
$$

Inserting this in the Friedmann-equation we get:

$$
\begin{aligned}
\dot{a} &= \sqrt{\frac{8G\pi\rho_0}{3}}\frac{1}{a} & (60)\\
da \cdot a &= \sqrt{\frac{8G\pi\rho_0}{3}} & (61)
\end{aligned}
$$

$$\left[\frac{1}{2}a^2\right]_0^t = \sqrt{\frac{8G\pi\rho_0}{3}}t \tag{62}$$

$$a(t) = \left[\frac{32G\pi\rho_0}{3}\right]^{\frac{1}{4}}\sqrt{t} \tag{63}$$

$$\frac{a(t)}{a(t_0)} = a(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2}} . \tag{64}$$

Thus we get for the time evolution of the density

$$\rho = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2}\,, \tag{65}$$

which is identical to the matter case. For the Hubble parameter we get

$$H = \frac{\dot{a}}{a} = \frac{1}{2}\frac{t^{-\frac{1}{2}}}{t_0^{\frac{1}{2}}}\frac{t_0^{\frac{1}{2}}}{t^{\frac{1}{2}}} = \frac{1}{2}\frac{1}{t}\,. \tag{66}$$

4.1.6 Solving the Equations for Matter and Radiation with $k = 0$

	Matter	Radiation
ρ	$\frac{\rho_o}{a^3}$	$\frac{\rho_o}{a^4}$
$a(t)$	$\left(\frac{t}{t_0}\right)^{\frac{2}{3}}$	$\left(\frac{t}{t_0}\right)^{\frac{1}{2}}$
ρ	$\frac{\rho_0 t_0^2}{t^2}$	$\frac{\rho_0 t_0^2}{t^2}$
$H =$	$\frac{2}{3}\frac{1}{t}$	$\frac{1}{2}\frac{1}{t}$

(67)

Having $\rho = \rho_{rad} + \rho_{matter}$ we can decouple the fluid equation

$$\begin{aligned} 0 &= \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) && (68) \\ &= \left[\dot{\rho}_{matter} + 3\frac{\dot{a}}{a}\rho_{matter}\right] + \left[\dot{\rho}_{rad} + 4\frac{\dot{a}}{a}\rho_{rad}\right] . && (69) \end{aligned}$$

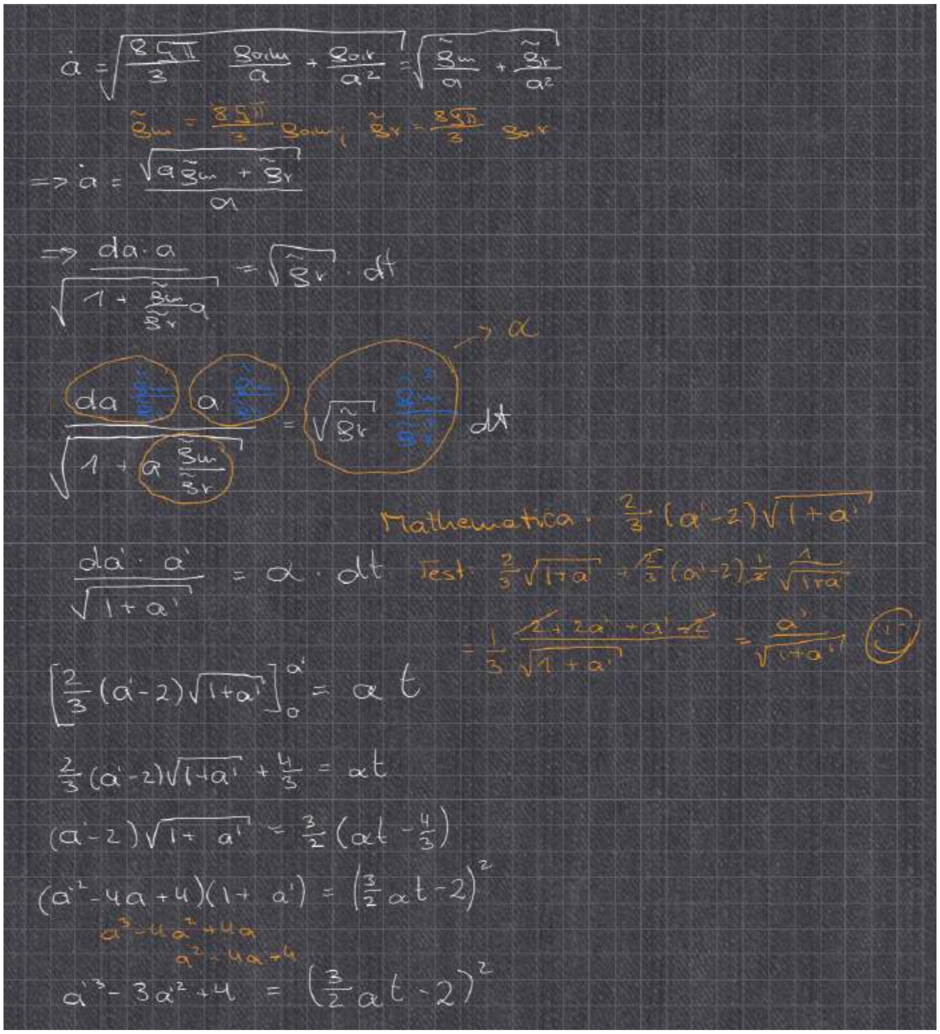
Thus we still have

$$\rho_{matter}(t) = \frac{\rho_{0,matter}}{a^3}\,, \qquad \rho_{rad}(t) = \frac{\rho_{0,rad}}{a^4}\,. \tag{70}$$

The Friedmann equations read thus

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}G\pi\left(\frac{\rho_{0,matter}}{a^3} + \frac{\rho_{0,rad}}{a^4}\right) \tag{71}$$

$$\Rightarrow \dot{a} = \sqrt{\frac{8}{3}G\pi\left(\frac{\rho_{0,matter}}{a} + \frac{\rho_{0,rad}}{a^2}\right)}\,. \tag{72}$$



$$\dot a = \sqrt{\frac{8\pi G}{3} \quad \frac{\rho_{0,m}}{a} + \frac{\rho_{0,r}}{a^2}} = \sqrt{\frac{\tilde\rho_m}{a} + \frac{\tilde\rho_r}{a^2}}$$

$$\tilde\rho_m = \frac{8\pi G}{3}\rho_{0,m}; \quad \tilde\rho_r = \frac{8\pi G}{3}\rho_{0,r}$$

$$\Rightarrow \dot a = \frac{\sqrt{a\tilde\rho_m + \tilde\rho_r}}{a}$$

$$\Rightarrow \frac{da\cdot a}{\sqrt{1+\frac{\tilde\rho_m}{\tilde\rho_r}a}} = \sqrt{\tilde\rho_r}\cdot dt$$

$$\frac{da \quad a}{\sqrt{1 + a\frac{\tilde\rho_m}{\tilde\rho_r}}} = \overset{\to\,\alpha}{\sqrt{\tilde\rho_r}} \quad dt$$

Mathematica: $\frac{2}{3}(a'-2)\sqrt{1+a'}$

$$\frac{da'\cdot a'}{\sqrt{1+a'}} = \alpha\cdot dt$$

Test: $\frac{2}{3}\sqrt{1+a'} + \frac{\cancel{2}}{3}(a'-2)\frac{1}{\cancel{2}}\frac{1}{\sqrt{1+a'}}$

$$= \frac{1}{3}\frac{\cancel{2} + 2a' + a' - \cancel{2}}{\sqrt{1+a'}} = \frac{a'}{\sqrt{1+a'}} \quad ☺$$

$$\left[\frac{2}{3}(a'-2)\sqrt{1+a'}\right]_0^{a'} = \alpha t$$

$$\frac{2}{3}(a'-2)\sqrt{1+a'} + \frac{4}{3} = \alpha t$$

$$(a'-2)\sqrt{1+a'} = \frac{3}{2}\left(\alpha t - \frac{4}{3}\right)$$

$$(a'^2 - 4a + 4)(1 + a') = \left(\frac{3}{2}\alpha t - 2\right)^2$$

$$a^3 - 4a^2 + 4a$$
$$a^2 - 4a + 4$$

$$a'^3 - 3a'^2 + 4 = \left(\frac{3}{2}\alpha t - 2\right)^2$$

The exact solution looks messy, but we can understand its properties, by considering the limiting cases: matter domination and radiation domination. If **radiation is dominant**, then we have $a(t) \propto t^{\frac{1}{2}}$, thus

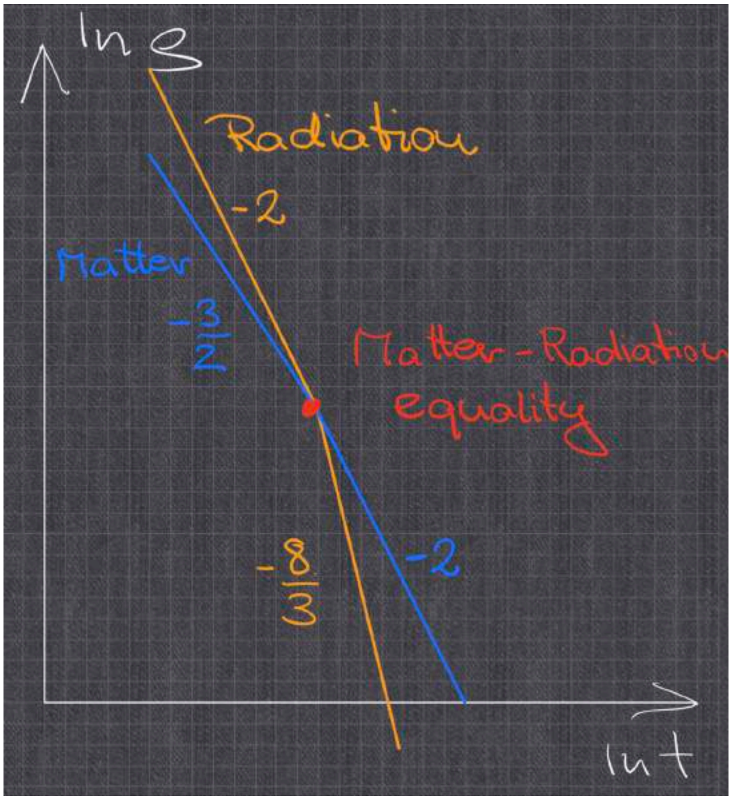
$$\rho_{rad} \propto \frac{1}{a^4} \quad \propto \quad \frac{1}{t^2} \Rightarrow \ln\rho_{rad} \propto -2\ln t\,, \tag{73}$$

$$\rho_{matter} \propto \frac{1}{a^3} \quad \propto \quad \frac{1}{t^{\frac{3}{2}}} \Rightarrow \ln \rho_{matter} \propto -\frac{3}{2} \ln t\,. \tag{74}$$

If **matter is dominant**, then we have $a(t) \propto t^{\frac{2}{3}}$, thus

$$\rho_{rad} \propto \frac{1}{a^4} \quad \propto \quad \frac{1}{t^{\frac{8}{3}}} \Rightarrow \ln \rho_{raf} \propto -\frac{8}{2} \ln t\,, \tag{75}$$

$$\rho_{matter} \propto \frac{1}{a^3} \quad \propto \quad \frac{1}{t^2} \Rightarrow \ln \rho_{matter} \propto -2 \ln t\,. \tag{76}$$



4.1.7 Particle Number Densities

The particle number density n is defined as

$$\epsilon = \rho c^2 =: nE\,. \tag{77}$$

n is useful when particle number is observed, e.g. in thermal equilibrium. In such cases n only changes with the volume and we get

$$n \propto \frac{1}{a^3}\,. \tag{78}$$

How does this comply with our earlier results?

1. Matter: for non-relativistic matter, E is constant and given by the mass, thus

$$\rho = n\frac{E}{c^2} \propto \frac{1}{a^3}\,. \tag{79}$$

2. Radiation: for radiation the energy drops of like $E \propto 1/a$, thus we get

$$\rho = n\frac{E}{c^2} \propto \frac{1}{a^4}\,. \tag{80}$$

This is a nice consistency check of our formulae.

4.1.8 Evolution including Curvature

The fluid equation is the same as above and the Friedmann-equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}G\pi\rho - \frac{k}{a^2}\,, \tag{81}$$

A first question that might come to ones mind is: under what circumstances does the evolution of the Universe stop?

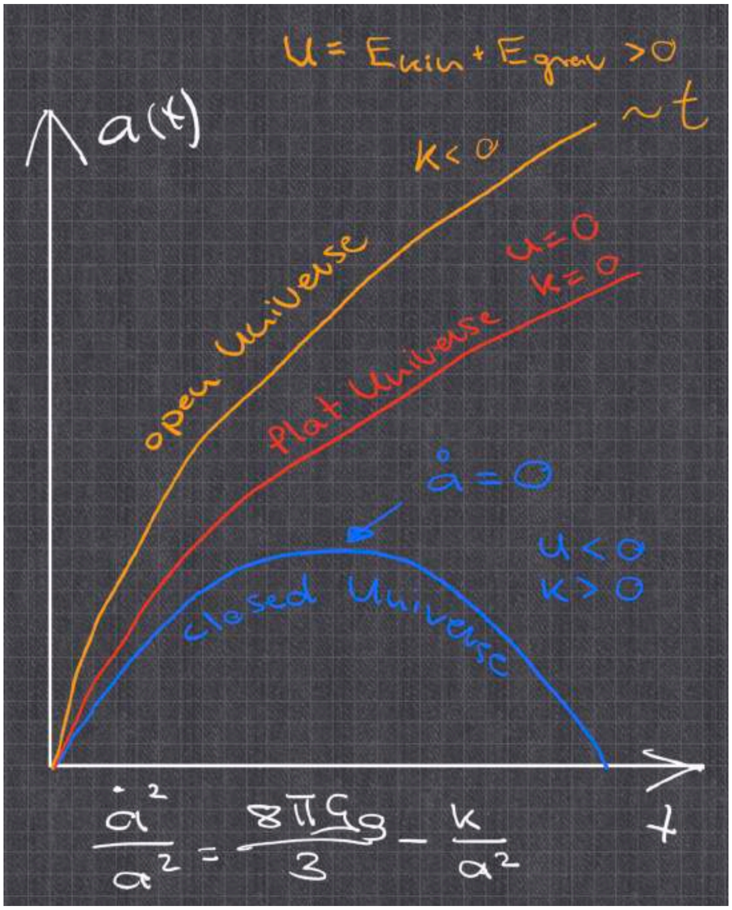
$$\dot{a} = 0 \;\Leftrightarrow\; \frac{8}{3}G\pi\rho = \frac{k}{a^2}\,. \tag{82}$$

Since ρ and a are positive, a stopping of the expansion can only happen for positive values of k, thus an open and a flat Universe expand forever.
Since $\rho \propto 1/a^{3...4}$ we get for open universes $k < 0$ for very large times

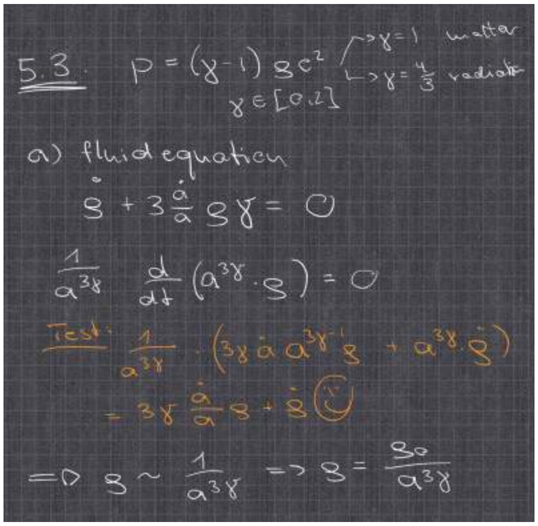
$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8}{3}G\pi\rho - \frac{k}{a^2} \approx -\frac{k}{a^2}\,, \tag{83}$$

$$\Rightarrow \dot{a} = \pm\sqrt{-k} \Rightarrow a = a_0 \pm \sqrt{-k}t\,. \tag{84}$$

In the case of a closed universe the expansion will stop and due to the persistence of the gravitational attraction, the Universe will contiue to collaps.


$U = E_{kin} + E_{grav} > 0$
a(t)
~t
k < 0
open Universe
U = 0
k = 0
Flat Universe
$\mathring{a} = 0$
U < 0
k > 0
closed Universe
t
$\frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho}{3} - \frac{k}{a^2}$

Example 5.3



5.3. $p = (\gamma - 1)\rho c^2$ $\nearrow \gamma = 1$ matter, $\searrow \gamma = \frac{4}{3}$ radiat

$\gamma \in [0,2]$

a) fluid equation

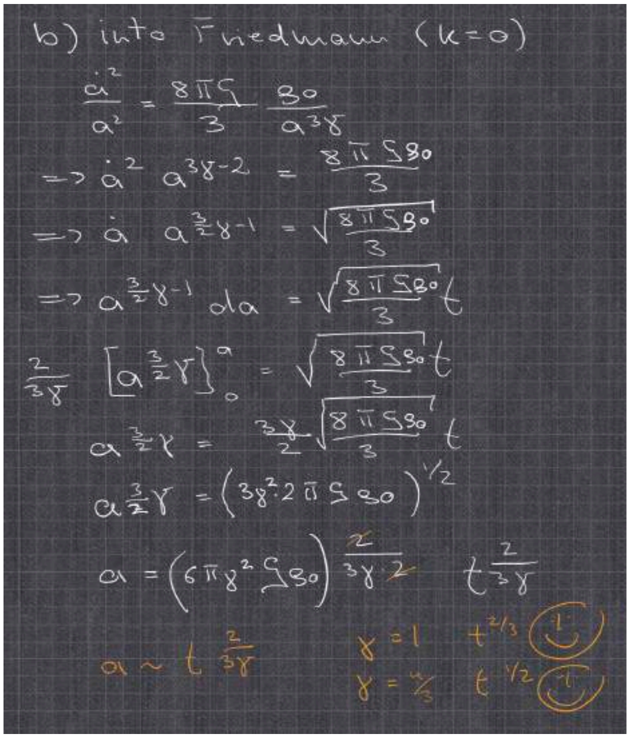
$$\dot{\rho} + 3\frac{\dot{a}}{a}\rho\gamma = 0$$

$$\frac{1}{a^{3\gamma}}\frac{d}{dt}\left(a^{3\gamma}\cdot\rho\right) = 0$$

Test: $\frac{1}{a^{3\gamma}}\cdot\left(3\gamma\dot{a}a^{3\gamma-1}\rho + a^{3\gamma}\dot{\rho}\right)$

$= 3\gamma\frac{\dot{a}}{a}\rho + \dot{\rho}$ ☺

$$\Rightarrow \rho \sim \frac{1}{a^{3\gamma}} \Rightarrow \rho = \frac{\rho_0}{a^{3\gamma}}$$



b) into Friedmann $(k=0)$

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\frac{\rho_0}{a^{3\gamma}}$$

$$\Rightarrow \dot{a}^2 a^{3\gamma-2} = \frac{8\pi G\rho_0}{3}$$

$$\Rightarrow \dot{a}\ a^{\frac{3}{2}\gamma-1} = \sqrt{\frac{8\pi G\rho_0}{3}}$$

$$\Rightarrow a^{\frac{3}{2}\gamma-1}\,da = \sqrt{\frac{8\pi G\rho_0}{3}}\,t$$

$$\frac{2}{3\gamma}\left[a^{\frac{3}{2}\gamma}\right]_0^a = \sqrt{\frac{8\pi G\rho_0}{3}}\,t$$

$$a^{\frac{3}{2}\gamma} = \frac{3\gamma}{2}\sqrt{\frac{8\pi G\rho_0}{3}}\,t$$

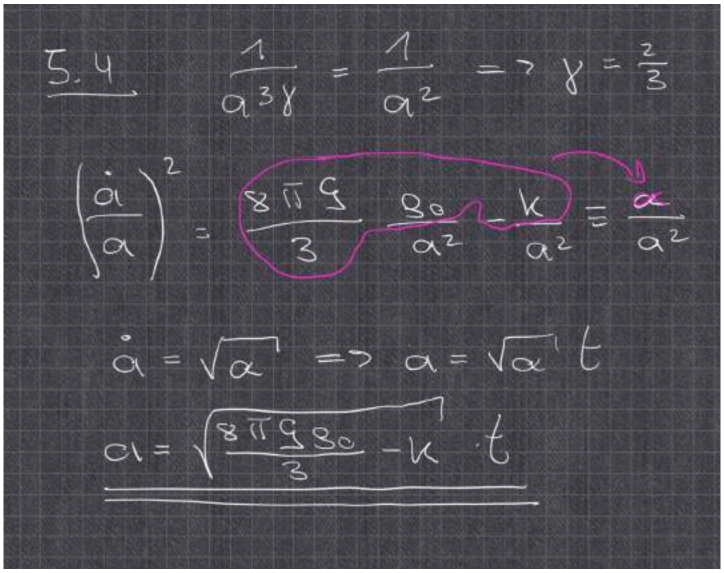
$$a^{\frac{3}{2}\gamma} = \left(3\gamma^2 2\pi G\rho_0\right)^{1/2}$$

$$a = \left(6\pi\gamma^2 G\rho_0\right)^{\frac{\cancel{2}}{3\gamma\cdot\cancel{2}}} t^{\frac{2}{3\gamma}}$$

$a \sim t^{\frac{2}{3\gamma}}$ $\gamma = 1$ $t^{2/3}$ ☺

$\gamma = \frac{4}{3}$ $t^{1/2}$ ☺

Example 5.4



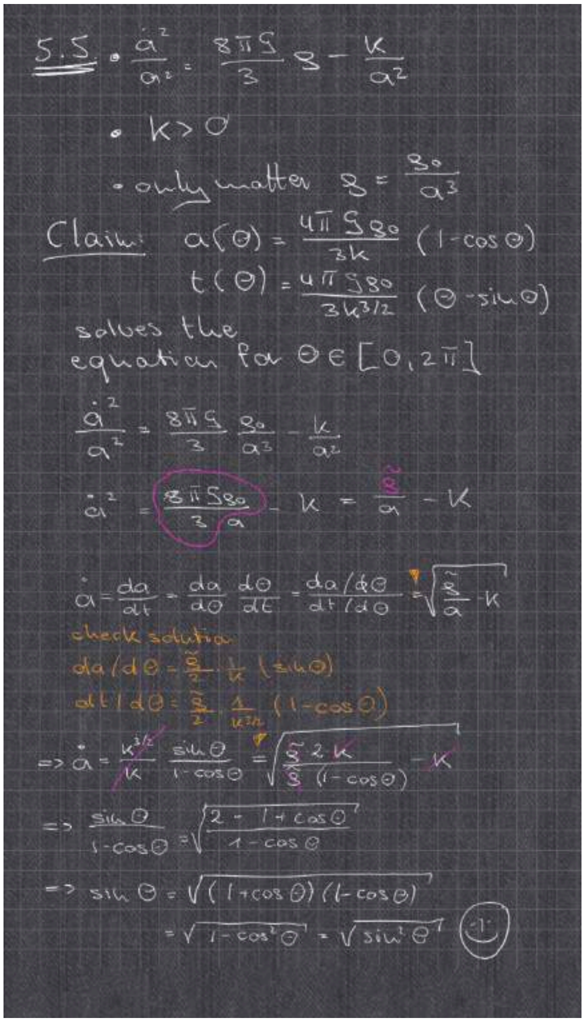
$\underline{5.4}$ $\quad \frac{1}{a^{3\gamma}} = \frac{1}{a^2} \Rightarrow \gamma = \frac{2}{3}$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^2} - \frac{k}{a^2} \equiv \frac{\alpha}{a^2}$$

$$\dot{a} = \sqrt{\alpha} \Rightarrow a = \sqrt{\alpha}\, t$$

$$\underline{\underline{a = \sqrt{\frac{8\pi G \rho_0}{3} - k} \cdot t}}$$

Example 5.5



5.5 • $\frac{\dot a^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$

• $k > 0$

• only matter $\rho = \frac{\rho_0}{a^3}$

Claim: $a(\Theta) = \frac{4\pi G\rho_0}{3k}(1-\cos\Theta)$

$t(\Theta) = \frac{4\pi G\rho_0}{3k^{3/2}}(\Theta-\sin\Theta)$

solves the equation for $\Theta \in [0, 2\pi]$

$$\frac{\dot a^2}{a^2} = \frac{8\pi G}{3}\frac{\rho_0}{a^3} - \frac{k}{a^2}$$

$$\dot a^2 = \boxed{\frac{8\pi G\rho_0}{3\,a}} - k = \frac{\tilde\rho}{a} - k$$

$$\dot a = \frac{da}{dt} = \frac{da}{d\Theta}\frac{d\Theta}{dt} = \frac{da/d\Theta}{dt/d\Theta} \overset{!}{=} \sqrt{\frac{\tilde\rho}{a} - k}$$

check solution

$da/d\Theta = \frac{\tilde\rho}{2}\frac{1}{k}(\sin\Theta)$

$dt/d\Theta = \frac{\tilde\rho}{2}\frac{1}{k^{3/2}}(1-\cos\Theta)$

$$\Rightarrow \dot a = \frac{\cancel{k^{3/2}}}{\cancel{k}}\frac{\sin\Theta}{1-\cos\Theta} \overset{!}{=} \sqrt{\frac{\cancel{\tilde\rho}\,2\,\cancel{k}}{\cancel{\tilde\rho}\,(1-\cos\Theta)} - \cancel{k}}$$

$$\Rightarrow \frac{\sin\Theta}{1-\cos\Theta} = \sqrt{\frac{2-1+\cos\Theta}{1-\cos\Theta}}$$

$$\Rightarrow \sin\Theta = \sqrt{(1+\cos\Theta)(1-\cos\Theta)}$$

$$= \sqrt{1-\cos^2\Theta} = \sqrt{\sin^2\Theta}$$

4.2 Observational Parameters

4.2.1 Hubble Parameter

The Hubble parameter can be written as

$$H_0 = 100\,h\,kms^{-1}Mpc^{-1}\,. \tag{85}$$

Current measurements give

$$\begin{aligned} H &= 73.52 \pm 1.62\,kms^{-1}Mpc^{-1} \text{ Cosmic Ladder}\,. && (86)\\ H &= 67.8 \pm 0.9\,kms^{-1}Mpc^{-1} \text{ CMB}\,. && (87)\\ \Rightarrow H &= 70.97 \pm 4.17\,kms^{-1}Mpc^{-1} \text{ Conservative}\,. && (88) \end{aligned}$$

4.2.2 The Density Parameter Ω_0

The Friedmann equation reads

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho - \frac{k}{a^2} && (89)\\ \Rightarrow -\frac{k}{a^2} &= H^2 - \frac{8\pi G}{3}\rho\,. && (90) \end{aligned}$$

For a given value of H, we can ask what would the density have to be in order to yield $k = 0$? We denote this value as the **critical density** ρ_c:

$$\begin{aligned} \rho_c &= \frac{3H^2}{8\pi G} && (91)\\ &= 1.88h^2 \cdot 10^{-26}\frac{kg}{m^3} = 9.5 \cdot 10^{-27}\frac{kg}{m^3} && (92)\\ &= 11.3h^2\frac{\text{protons}}{m^3} \approx 5.7\frac{\text{protons}}{m^3} && (93)\\ &= 2.78h^2 \cdot 10^{11}\frac{M_\odot}{Mpc^3} = 1.4 \cdot 10^{11}\frac{M_\odot}{Mpc^3}\,. && (94) \end{aligned}$$

The **density parameter** $\Omega(t)$ is defined as

$$\Omega(t) = \frac{\rho}{\rho_c}\,, \tag{95}$$

the present value of Ω is denoted as Ω_0. Thus we can rewrite the Friedmann equation

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho_c\Omega - \frac{k}{a^2} = H^2\Omega - \frac{k}{a^2} && (96)\\ \Rightarrow 1 &= \Omega - \frac{k}{H^2a^2} =: \Omega + \Omega_k\,. && (97) \end{aligned}$$

4.2.3 The Deceleration Parameter q_0

Do a Taylor expansion of the scale factor

$$\begin{aligned} a(t) &= a(t_0) + \dot{a}(t_0)(t-t_0) + \ddot{a}(t_0)(t-t_0)^2 + \dots && (98) \\ \frac{a(t)}{a(t_0)} &= 1 + \frac{\dot{a}(t_0)}{a(t_0)}(t-t_0) + \frac{1}{2}\frac{\ddot{a}(t_0)}{a(t_0)}(t-t_0)^2 + \dots && (99) \\ &= 1 + H_o(t-t_0) - \frac{q_0}{2}H_0^2(t-t_0)^2 + \dots\,, && (100) \end{aligned}$$

with the decelaeration parameter

$$q_0 = -\frac{\ddot{a}(t_0)}{a(t_0)}\frac{1}{H_0^2} = -\frac{\ddot{a}(t_0)a(t_0)}{\dot{a}^2(t_0)}\,. \qquad (101)$$

Remember: the acceleration equation:

$$\begin{aligned} \frac{\ddot{a}}{a} &= -4G\pi\left(\frac{\rho}{3} + \frac{p}{c^2}\right) && (102) \\ \Rightarrow q_0 = -\frac{1}{H_0^2}\frac{\ddot{a}_0}{a_0} &= \frac{4G\pi}{H_0^2}\left(\frac{\rho_0}{3} + \frac{p_0}{c^2}\right) && (103) \\ &= \frac{1}{2}\frac{8G\pi}{3H_0^2}\left(\rho_0 + 3\frac{p_0}{c^2}\right) = \frac{1}{2}\frac{1}{\rho_c}\left(\rho_0 + 3\frac{p_0}{c^2}\right) && (104) \\ &= \frac{\Omega_0}{2} + \frac{3}{2}\frac{p_0}{\rho_c c^2}\,. && (105) \end{aligned}$$

Thus we get for a matter dominated Universe ($p = 0$)

$$q_0 = \frac{\Omega_0}{2}\,. \qquad (106)$$

BUT: q_0 was measured with a negative value!!!!

5 Lecture 4: Modelling the Universe III

We 21.11.: Chapter 7,8

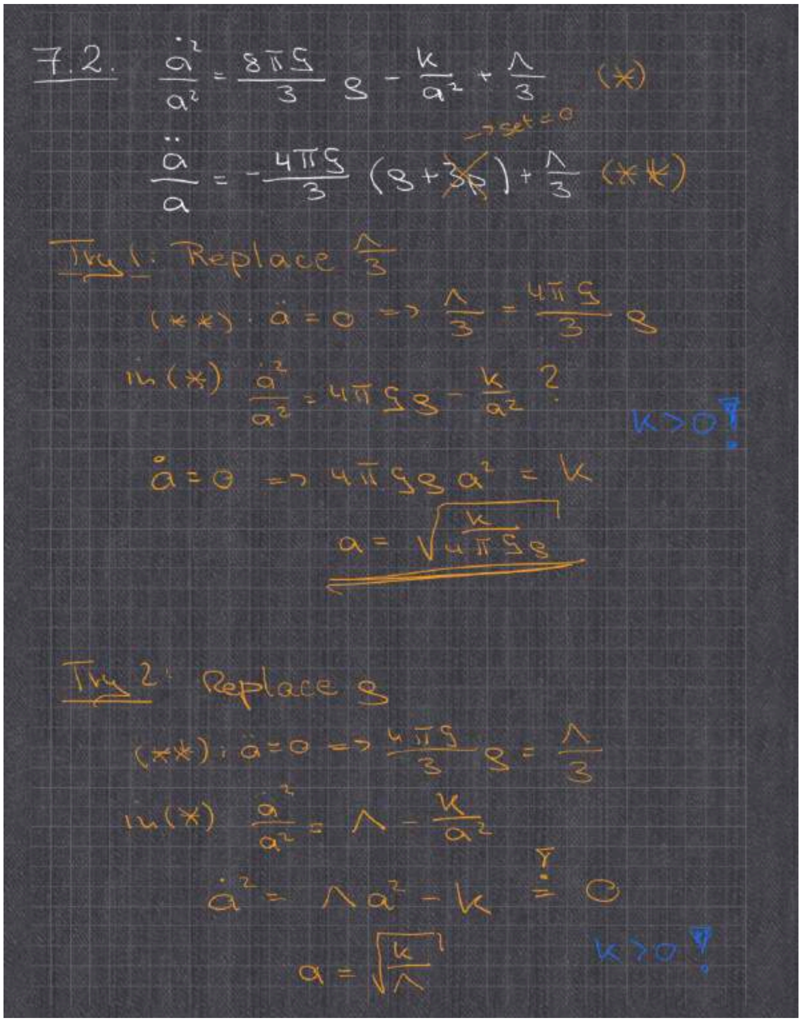
5.1 The Cosmological Constant

5.1.1 Definition

Originally introduced by Einstein to get static solutions, which actually did not exist. Later he called this his "greatest blunder". GTR allows a cosmological constant.
2011 Nobel prize: Perlmutter, Schmidt, Riess
The Friedmann Equation including a cosmological constant reads

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \,. \tag{107}$$

- Λ is measured in [time]$^{-2}$.
- Λ can be positive or negative, although mostly the positive case is considered
- Original idea: Λ should balance k in order to get $H = 0$ (See Example 7.2). **But** such a constellation will be unstable to small perturbations.
  Example 7.2



7.2. $$\frac{\dot a^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \quad (*)$$

$$\frac{\ddot a}{a} = -\frac{4\pi G}{3}(\rho + \overset{\to \text{set} = 0}{\cancel{3p}}) + \frac{\Lambda}{3} \quad (**)$$

Try 1: Replace $\frac{\Lambda}{3}$

$(**)$: $\ddot a = 0 \Rightarrow \frac{\Lambda}{3} = \frac{4\pi G}{3}\rho$

in $(*)$ $\frac{\dot a^2}{a^2} = 4\pi G\rho - \frac{k}{a^2}$?

$k > 0$!

$\dot a = 0 \Rightarrow 4\pi G\rho a^2 = k$

$$a = \sqrt{\frac{k}{4\pi G\rho}}$$

Try 2: Replace ρ

$(**)$: $\ddot a = 0 \Rightarrow \frac{4\pi G}{3}\rho = \frac{\Lambda}{3}$

in $(*)$ $\frac{\dot a^2}{a^2} = \Lambda - \frac{k}{a^2}$

$$\dot a^2 = \Lambda a^2 - k \overset{!}{=} 0$$

$$a = \sqrt{\frac{k}{\Lambda}}$$

$k > 0$!

- Nowadays: $\Lambda > 0$, $k = 0$ favoured.

Following the same steps ($\dot{\Lambda} = 0$) as in the second lecture we can derive the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \,. \tag{108}$$

A positive cosmological constant gives a positive contribution to $\ddot{a}$ and acts thus as a kind of repulsive force. If it is large enough it can overcome the first term of the acceleration equation, originating from density and pressure.
Defining

$$\Omega_\Lambda = \frac{\Lambda}{3H^2} \,, \tag{109}$$

we can rewrite the Friedmann equation as

$$\Omega + \Omega_\Lambda + \Omega_k = 1 \,. \tag{110}$$

5.1.2 Fluid Description of Λ

Sometimes it can be helpful to describe Λ as a fluid with density ρ_Λ and pressure p_Λ.
Defining

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \qquad \left(\Rightarrow \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c}\right) \,, \tag{111}$$

we can rewrite the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}(\rho + \rho_\Lambda) - \frac{k}{a^2} \,. \tag{112}$$

Since ρ_Λ is constant by definition we change in our usual equations (Friedmann and Acceleration) $\rho \to \rho + \rho_\Lambda$, $p \to p + p_\Lambda$ and use

$$\dot{\rho}_\Lambda + 3\frac{\dot{a}}{a}(\rho_\Lambda + p_\Lambda) = 0 \,, \tag{113}$$

$$\text{with} \qquad p_\Lambda = -\rho_\Lambda \,. \tag{114}$$

The negative pressure means that during the exansion work is done on the cosmological constant field. Thus its energy density can stay constant, despite the volume increase.
Then the acceleration equation reads

$$\begin{aligned}\frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}\left(\rho + \rho_\Lambda + 3(p + p_\Lambda)\right) . & (115)\\ &= -\frac{4\pi G}{3}\left(\rho - 2\rho_\Lambda + 3p\right) & (116)\\ &= -\frac{4\pi G}{3}\left(\rho + 3p\right) + \frac{8\pi G}{3}\rho_\Lambda & (117)\\ &= -\frac{4\pi G}{3}\left(\rho + 3p\right) + \frac{\Lambda}{3} & (118)\end{aligned}$$

The nature of the cosmological constant is unclear.

- It might be a kind of energy of the empty space, particle theories are predicting such a quantity, but with a value that is a factor of 10^{120} larger than the cosmological observation. = **cosmological constant problem**

- The cosmological constant, might also be a temporarily phenomenon, which disappears in the future.

- It could be **quintessence**, i.e. small variations are possible in the constant. The quintessence fluid could have the following equation of state

  $$p_Q = \omega \rho_Q \,. \qquad (119)$$

  $\omega = -1$ corresponds to the cosmological constant case, while accelerated expansion is possible for $\omega < -1/3$.

5.1.3 Cosmological Models with Λ

The unexpected discovery of a cosmological constant has forced physicists to rethink some of the standard lore!
Greatly increases the possible behaviors of the Universe:

- closed universes do not necessarily recollapse.

- open universes do not necessarily expand forever.

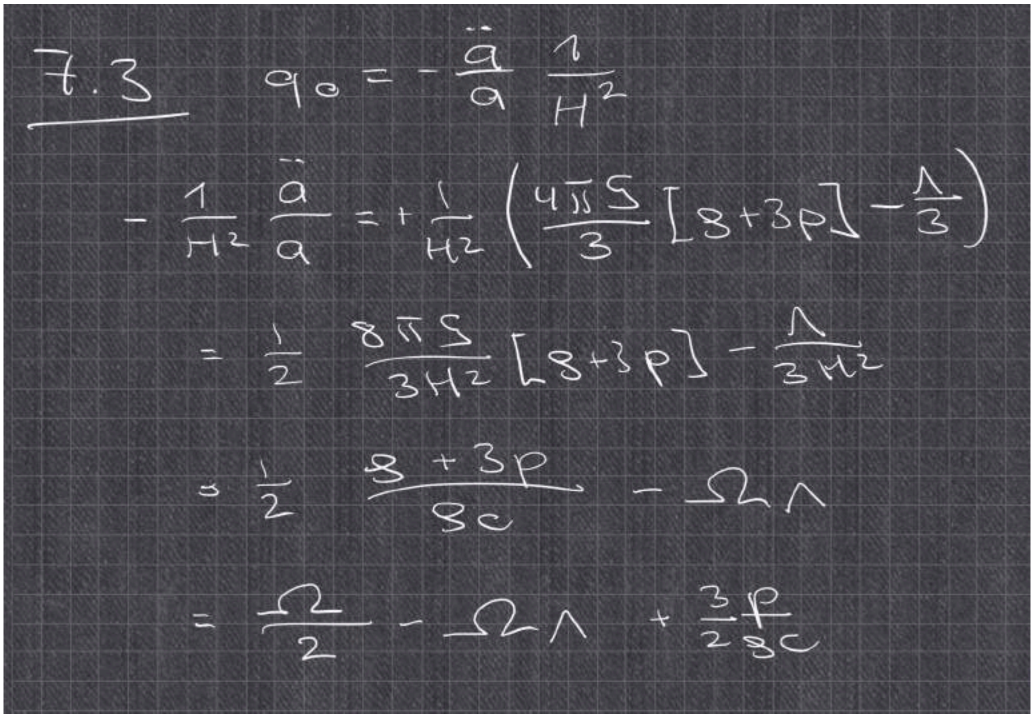
- if the cosmological constant is large enough, there would even be no need for a Big Bang - the Universe could start with a collapsing phase, followed by a bounce at finite size due to Λ (ruled out by observation).
- There could be a phase of an almost static Universe (**loitering**).

Look into the Ω_0-Ω_Λ-plane and assume that the matter in the present Universe is pressureless.

- $\Omega_0 + \Omega_\Lambda = 1$ gives a flat Universe.
- One can show that the deceleration parameter q_0 reads now

$$q_0 = \frac{\Omega_0}{2} - \Omega_\Lambda\,. \tag{120}$$

Example 7.3



7.3

$$q_0 = -\frac{\ddot{a}}{a}\frac{1}{H^2}$$

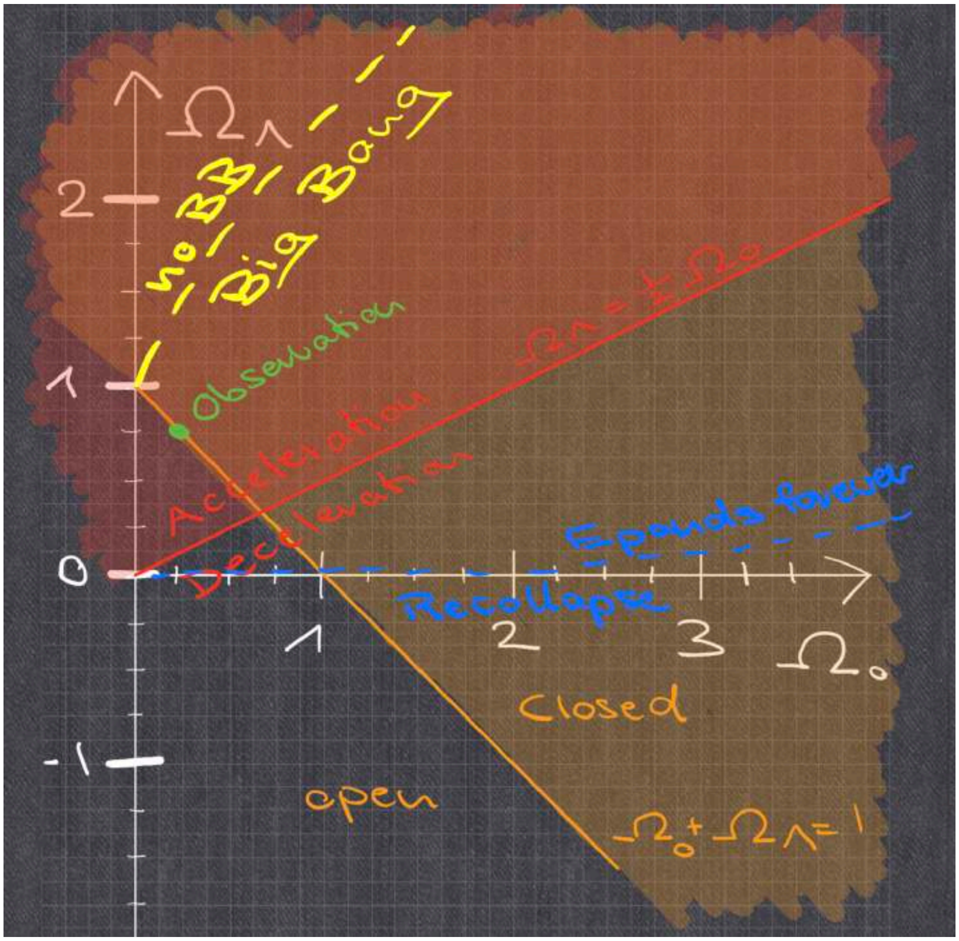
$$-\frac{1}{H^2}\frac{\ddot{a}}{a} = +\frac{1}{H^2}\left(\frac{4\pi G}{3}[\varrho+3p] - \frac{\Lambda}{3}\right)$$

$$= \frac{1}{2}\frac{8\pi G}{3H^2}[\varrho+3p] - \frac{\Lambda}{3H^2}$$

$$= \frac{1}{2}\frac{\varrho+3p}{\varrho_c} - \Omega_\Lambda$$

$$= \frac{\Omega}{2} - \Omega_\Lambda + \frac{3p}{2\varrho_c}$$

The fate of the Universe



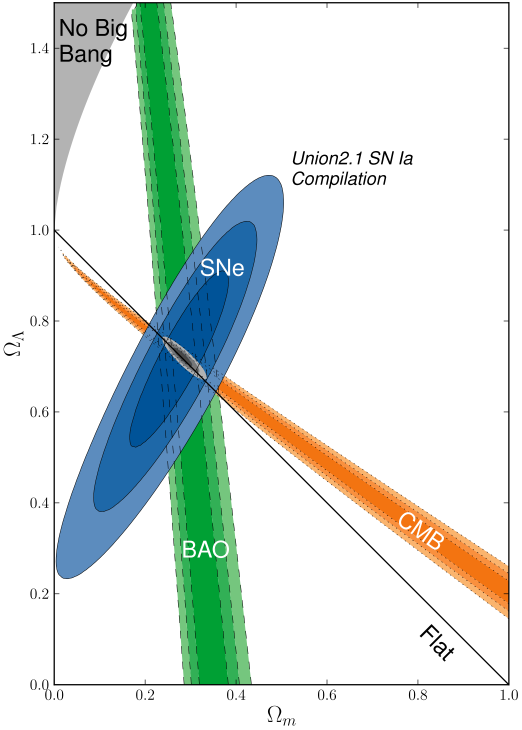
Current values of the observational parameters (from PDG)

$$\Omega_{Matter} = 0.306 \pm 0.007\,, \tag{121}$$

$$\Omega_\Lambda = 0.694 \pm 0.007\,, \tag{122}$$

$$\Omega = 1.0002 \pm 0.0026\,, \tag{123}$$

$$\omega = -1.01 \pm 0.04\,. \tag{124}$$



Supernova Cosmology Project
Suzuki, et al., *Ap.J.* (2011)

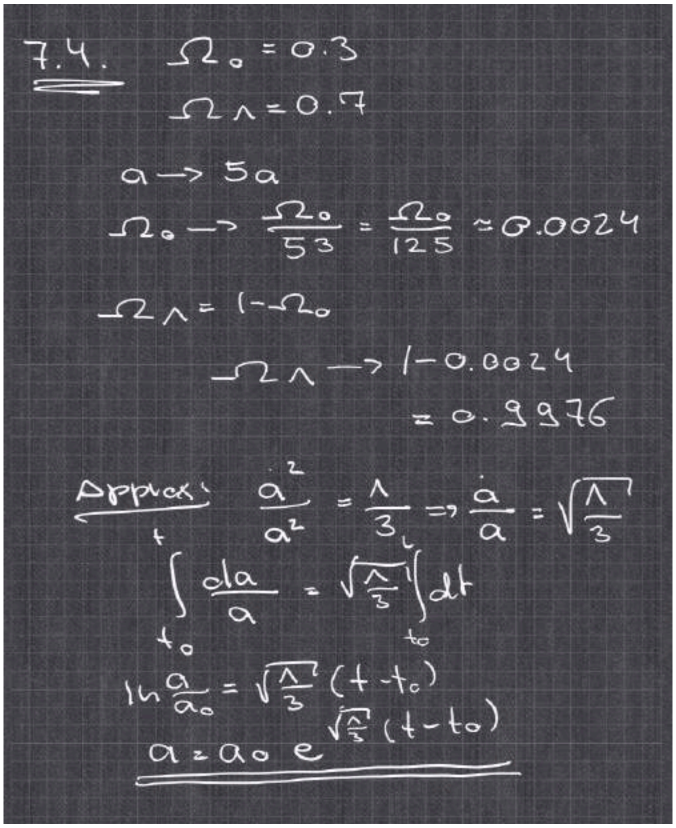
Baryon acoustic oscillations (BAO) are fluctuations in the density of the visible baryonic matter (normal matter) of the universe, caused by acoustic density waves in the primordial plasma of the early universe.

BAO matter clustering provides a ”standard ruler” for length scale in cosmology.

The length of this standard ruler is given by the maximum distance the acoustic waves could travel in the primordial plasma before the plasma cooled to the point where it became neutral atoms (the epoch of recombination), which stopped the expansion of the plasma density waves, ”freezing” them into place.

The length of this standard ruler (about 490 million light years in today’s universe) can be measured by looking at the large scale structure of matter using astronomical surveys.

Example 7.4



7.4. $\Omega_0 = 0.3$

$\Omega_\Lambda = 0.7$

$a \rightarrow 5a$

$\Omega_0 \rightarrow \frac{\Omega_0}{5^3} = \frac{\Omega_0}{125} \approx 0.0024$

$\Omega_\Lambda = 1 - \Omega_0$

$\Omega_\Lambda \rightarrow 1 - 0.0024$

$= 0.9976$

Approx: $\frac{\dot{a}^2}{a^2} = \frac{\Lambda}{3} \Rightarrow \frac{\dot{a}}{a} = \sqrt{\frac{\Lambda}{3}}$

$\int_{t_0}^{t} \frac{da}{a} = \sqrt{\frac{\Lambda}{3}} \int_{t_0}^{t} dt$

$\ln \frac{a}{a_0} = \sqrt{\frac{\Lambda}{3}} (t - t_0)$

$a = a_0 e^{\sqrt{\frac{\Lambda}{3}} (t - t_0)}$

5.2 The age of the Universe

What is the age of the Universe t_0?
Can be predicted from our cosmological models and then compared with observational evidence.
Some approximations first: if the Universe would have always exapnded with the same velocity as it is expanding now, then the age of the universe would be the **Hubble time**

$$t_H = \frac{1}{H_0} = \frac{9.77}{h} \cdot 10^9 \, years = 13.78 \cdot 10^9 \, years \,. \qquad (125)$$

Different indications for the age of the universe:

- Age of Earth: 5 billion years.
- Uranium isotopes in the galactic disc: 10 billion years.
- Old globular clusters: 10-13 (+1 for forming) billion years.

The rough agreement of these values is a strong confirmatin of the Big Bang idea.
Let us look a little closer to our equations: if the Universe is matter dominated and flat, then we got

$$\begin{aligned} H &= \frac{2}{3}\frac{1}{t} && (126) \\ \Rightarrow H_0 &= \frac{2}{3}\frac{1}{t_0} && (127) \\ \Rightarrow t_0 &= \frac{2}{3}\frac{1}{H_0} = \frac{6.51}{h} \cdot 10^9 \, years = 9.2 \cdot 10^9 \, years \, . && (128) \end{aligned}$$

This starts to become troublesome. If $\Omega_0 > 0$ then it becomes even worse! What is the age of a flat Universe with a cosmological constant?

$$\begin{aligned} H_0 t_0 &= \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}} \ln\left[\frac{1+\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right] && (129) \\ &= \frac{2}{3}\frac{1}{\sqrt{1-\Omega_0}} \sinh^{-1}\left[\frac{\sqrt{1-\Omega_0}}{\sqrt{\Omega_0}}\right] \, . && (130) \end{aligned}$$

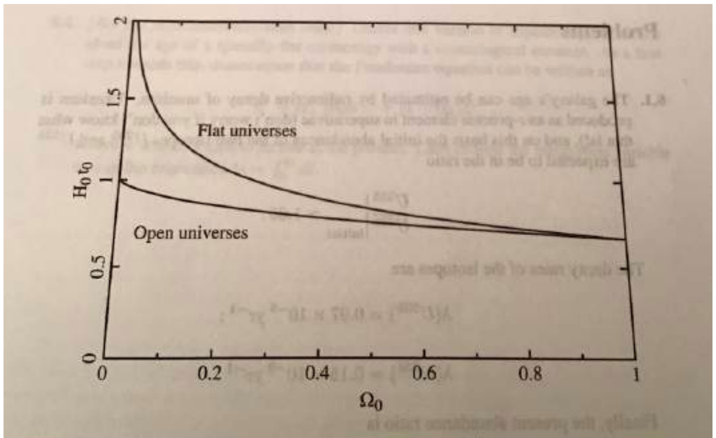


Figure 8.1 Predicted ages as fractions of the Hubble time H_0^{-1}, for open universes and for universes with a flat geometry plus a cosmological constant. The prediction $H_0 t_0 = 2/3$ for critical-density models is at the right-hand edge.

- We get $H_0 t_0 = 1$ for $\Omega_0 = 0.26$.
- For the observed value of $\Omega_0 = 0.306$ we get

$$\begin{aligned} H_0 t_0 &= 0.958767 && (131) \\ t_0 &= \frac{0.958767}{H_0} = 13.2\,Gyr\,. && (132) \end{aligned}$$

This can be compared to CMB investigations yielding

$$t_0 = 13.80 \pm 0.04\,Gyr\,. \qquad (133)$$

6 Lecture 5: The Dark Side of the Universe

Mo 26.11.: Chapter 9,10

6.1 The Density of the Universe and Dark Matter

What is the value of Ω_0 in the Universe?
How is Ω_0 composed?

Background information in:
History of dark matter, G. Bertone and D. Hooper,
Rev. Mod. Phys. **90** (2018) no.4, 045002; arXiv:1605.04909 astro-ph
111 citations counted in INSPIRE as of 23 Nov 2018

6.1.1 Weighing the Universe

Remember:

$$\begin{aligned}\rho_c &= 1.88h^2 \cdot 10^{-27}\frac{kg}{m^3} && (134)\\ &= 2.78h^2 \cdot 10^{11}\frac{M_\odot}{Mpc^3}\,. && (135)\end{aligned}$$

1. **Counting Stars**

$$\Omega_{Stars} = \frac{\rho_{stars}}{\rho_c} \approx 0.005 \longrightarrow 0.01\,. \quad (136)$$

Not all of the material in the Universe is in shinning stars; e.g. large amounts of gas, faint low-mass stars, like brown dwarfs (Jupiters, $< 0.08M_\odot$).
The Hubble telsecope did not find a sbustantial amount of brown dwarfs - so they seem not to be very important.

2. **Nucleosynthesis**
Theoretical calculation of the abundance of elements in the universe depends crucially on the baryonic matter density. The predictions agree with observation, if

$$0.021 \leq \Omega_B h^2 \leq 0.025 \Rightarrow \begin{cases} \Omega_B \leq \frac{0.025}{h^2} = \frac{0.025}{(0.7097-0.0417)^2} = 0.056\,, \\ \Omega_B \geq \frac{0.021}{h^2} = \frac{0.021}{(0.7097+0.0417)^2} = 0.037\,. \end{cases} \quad (137)$$

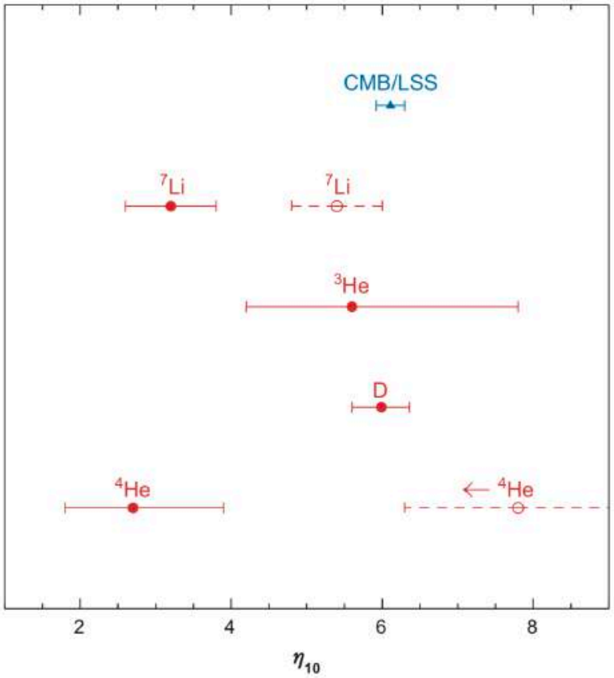


Figure 13

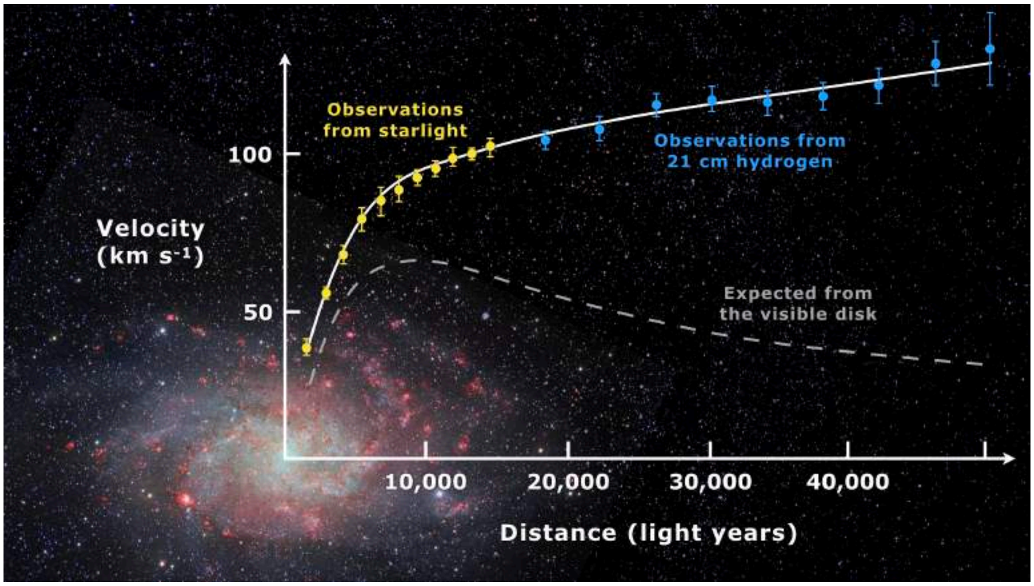
The SBBN-predicted values of η_{10}, and their 1σ uncertainties (*red filled circles*), corresponding to the primordial abundances adopted in Section 3.5, and the non-BBN value inferred from cosmic microwave background radiation (CMB) and large scale structure (LSS) data (*blue triangle*). The open circles and dashed lines correspond to the alternate abundances proposed for ^{4}He and ^{7}Li in Section 3.5.

3. **Galaxy rotation curves** 1940 Oort, 1957 Henk van de Hulst et al, 1959 Louise Volders , lates 1960's and 1970 Vera Rubin.

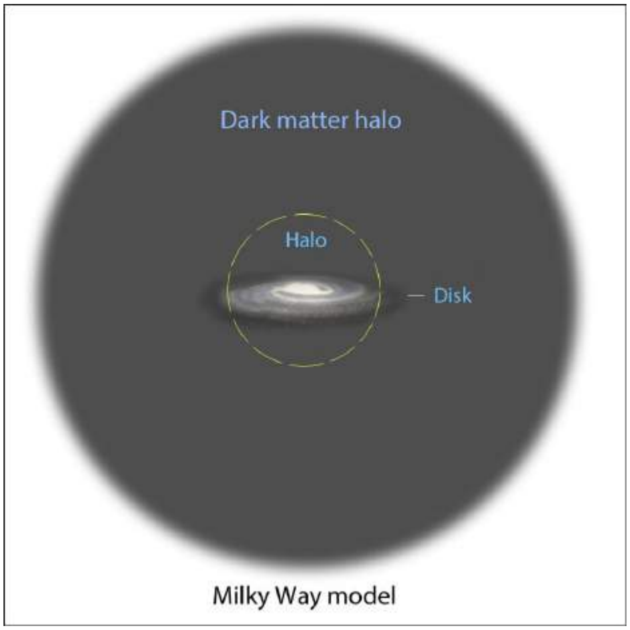
   Rotation curves of galaxies:

$$\frac{v^2}{R} = \frac{GM(R)}{R^2} \tag{138}$$

$$v = \sqrt{\frac{GM(R)}{R}} = \begin{cases} \sqrt{G\frac{4}{3}\pi\rho_0} \cdot R & \text{inside homogeneous matter distribution} \\ \sqrt{G\frac{4}{3}\pi\rho(R)}R \propto 1 & \text{inside dark matter halo with } \rho \propto \frac{1}{R^2} \\ \sqrt{GM}\frac{1}{\sqrt{R}} & \text{outside matter distribution} \end{cases} \tag{139}$$



So there seems to be a substantial amount of matter outside the visible disc of a galaxy. Detailed studies of rotation curves give the following picture



The overall amount of mater in the halo is estimated to be around

$$\Omega_{Halo} \approx 0.1\,. \tag{140}$$

4. **Galaxy Cluster Composition** 1933 Zwicky
   In clusters there is about 5 to 10 times as much gas as there are visible stars, e.g. CHANDRA X-ray satellite:

   $$\Omega_{cluster} \approx 0.4\,. \tag{141}$$

   Nowadays measurements are often done via gravitational lensing (strong and weak lensing).

5. **Structure Formation**
   Numerical simulations for the formulation of structures in the Universe, require a sizeable dark matter component in order to be in agreement with observation.
   Small initial irregularities have to form into the observed structures like galaxies, clusters,...
   The ICC is specialised on these simulations!

   $$\Omega_{structure} \approx 0.3\,. \tag{142}$$

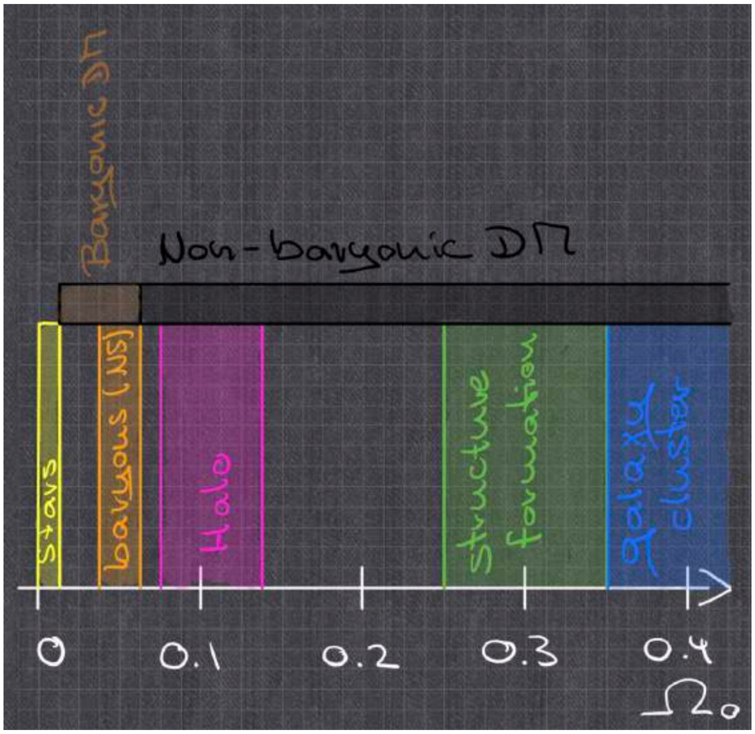
6. **Geometry of the Universe and Brightness of Supernovae**
   CMB and acceleration of the Universe... overall fit gives:

   $$\Omega_{Matter} = 0.306 \pm 0.007\,, \tag{143}$$
   $$\Omega_{\Lambda} = 0.694 \pm 0.007\,, \tag{144}$$
   $$\Omega = 1.0002 \pm 0.0026\,. \tag{145}$$

7. **Overview**

   $$\Omega_{stars} = 0.005...0.01\,, \tag{146}$$
   $$\Omega_{B} = 0.037...0.056\,, \tag{147}$$
   $$\Omega_{Halo} \approx 0.1\,, \tag{148}$$
   $$\Omega_{cluster} \approx 0.4\,, \tag{149}$$
   $$\Omega_{structure} \approx 0.3\,. \tag{150}$$

   This means there must be a sizable amount of **baronic dark matter** and **non-baronic dark matter**.
   The baryonic DM might be gas. There is about 5 times as much non-baryonic DM as baryonic matter.



The cosmological constant makes the largest contribution to the total density, which is extremely close to 1!

6.1.2 What might the Dark Matter be?

Ultimate Copernican viewpoint: we are not only in no special place in the Universe, we are even not made out of the most abundant stuff in the Universe.

1. **Fundamental Particles**
   Dark matter might consist of new fundamental particles:
   - In the SM: neutrinos, but this would be hot dark matter and in conflict with structure formation.
   - Beyond the SM:
     - Heavy neutrinos, e.g. sterile ones.

    - Lightest Supersymmetric Particle - in SUSY theories.
    - or more general: **WIMPs** = weakly interacing massive particles.

2. **Compact Objects**
    - **Balck Holes** primordial black holes that formed before nucleosynthesis.
    - **Machos** (Massive compact halo objects) like brown dwarfs, but also exotic non-baryonic objects...

   The expected average DM density is

$$\rho_{DM} \quad = \quad 0.3\rho_c = \frac{2 \text{ proton masses}}{m^3} \, . \tag{151}$$

   In galaxies this density could of course be considerably higher.

6.1.3 Dark Matter searches

Gravitational lensing can be used for compact objects, but not for elementary particles.
For elementary particles the worst case scenario would be if DM couples only gravitationally to the SM.
If there is some kind of weak interaction with the SM particles, either directly or via a messanger sector then we can do 3 kinds of searches:

1. Direct searches: like super CDMS - we have a replica of the DM detector for outreach.
2. Indirect searches.
3. Collider searches.

Check also:

Yet Another Introduction to Dark Matter,
M. Bauer and T. Plehn
arXiv:1705.01987 [hep-ph].

6.2 The Cosmic Microwave background

More detailed in *Cosmology II*

6.2.1 Properties of the CMB

Balck body radiation with a temperature of

$$T = 2.725 \pm 0.001 \, K \,. \tag{152}$$

The corresponding energy density is given by

$$\begin{aligned} \epsilon &= \rho_{rad} c^2 = \alpha T^4 & (153) \\ & \left(\alpha = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} = 7.565 \cdot 10^{-16} \frac{J}{m^3 K^4} \right) & (154) \\ &= 4.17 \cdot 10^{-14} \frac{J}{m^3} \,. & (155) \\ \Rightarrow \Omega_{rad} = \frac{\rho_{rad}}{\rho_c} &= 4.9 \cdot 10^{-5} \,. & (156) \end{aligned}$$

We know how the energy density of radiation falls off during the expansion of the universe:

$$\begin{aligned} \rho_{rad} &\propto \frac{1}{a^4} \,, & (157) \\ \alpha T^4 &\propto \frac{1}{a^4} \,, & (158) \\ \Rightarrow T &\propto \frac{1}{a} \,. & (159) \end{aligned}$$

Thus at an early stage of the Universe its temperature was much higher!
The energy density ϵ (energy per unit volume) in a frequency interval df around f is given by

$$\epsilon(f) df = \frac{8\pi h}{c^3} \frac{f^3 df}{\exp\left[\frac{hf}{k_b T}\right] - 1} \,. \tag{160}$$

During the expansion of the Universe, the frequency f will be reduced via $f \propto 1/a$. So the numerator $f^3 df$ will scale like $1/a^4$ - like the expected scaling for radiation, the denominator will stay constant as f/T is not changing during expansion. The temperature behaves as

$$T_{final} = T_{initial}\frac{a_{initial}}{a_{final}} . \tag{161}$$

The form of the distribution will thus stay also for times, when the photons are not in thermal equilibrium anymore.

6.2.2 The Photon to Baryon Ratio

If interactions are negligible, then particles cannot simply disappear. During the expansion of the Universe the number of baryons and the number of photons stays constant and we have

$$n \propto \frac{1}{a^3} . \tag{162}$$

Thus, the number densities of protons and photons will drop with the volume increase. What is the photon to proton ratio?
The present energy density of the CMB is

$$\epsilon_{rad}(t_0) = 4.17 \cdot 10^-14\frac{J}{m^3} . \tag{163}$$

The typical energy of a CMB photon is

$$E_{mean} = 3k_BT = 7.0 \cdot 10^{-4} eV . \tag{164}$$

Thus we get a photon number density of

$$n_\gamma = \frac{\epsilon_{rad}(t_0)}{E_{mean}} = 3.7 \cdot 10^8 \frac{1}{m^3} . \tag{165}$$

For the baryonic number density we get

$$n_B = \Omega_B\rho_c = 0.045 \cdot 5.7\frac{\text{protons}}{m^3} = 0.26\frac{\text{protons}}{m^3} . \tag{166}$$

Thus we get for the photon to baryon ratio

$$\frac{n_\gamma}{n_B} = \frac{3.7 \cdot 10^8 \frac{1}{m^3}}{0.26\frac{1}{m^3}} = 1.4 \cdot 10^9 . \tag{167}$$

6.2.3 The Origin of the CMB

Consider a time, when the Universe was a factor of 10^6 smaller

$$a \rightarrow a/10^6 \,, \tag{168}$$

$$T \rightarrow 10^6 T = 3 \cdot 10^K \,, \tag{169}$$

$$E_\gamma = 700 eV \,, \tag{170}$$

A typical photon had an energy that is much higher than the ionisation energy of Hydrogen (13.6 eV). A this stage we had a plasma of positively charged protons and negativly charged electrons. The photons interacted strongly with the free electrons via Thomson scattering, thus the Universe was more or less opaque.

During the expansion, the Universe cooled to temperatures well below 13.6 eV and the electrons and protons formed neutral Hydrogen, which only weakly interacts with the photons. Thus the photons **decouple** and the Universe becomes transparent.

When did the decoupling happen?

a) Simple estimate:

$$T \approx \frac{13.6 eV}{3 k_B} = 52590 K \,. \tag{171}$$

($k_B = 8.62 \cdot 10^{-5} \frac{eV}{K}$.

b) More precise: we have 10^9 photons per proton, and we have a high energy tail of the Boltzmann distribution

$$T \approx \frac{13.6 eV}{k_B ln(1.4 \cdot 10^9)} = 7400 K \,. \tag{172}$$

c) Even more precise: integrate over Boltzmann distribution: $5700K$. (Problem 10.5)

d) More or less exact: 3000 K - thus decoupling happened when the Universe was a factor 1000 smaller than now! The Universe had an age of about 350 000 years.

Surface of last scattering: sphere in a distance of about $6000 \cdot h^{-1}$ Mpc.

6.2.4 The Origin of the CMB II

More precisely, following Kolb and Turner.

1. There are two seperate processes going on:

   - **Recombination**: electrons join nuclei to form neutral atoms.
   - **Decoupling**: photons will not scatter again.

   If recombination would be instantaneous and complete, then both would coincide.
   In practice each process takes time and decpoupling follows recombination.

2. **Saha equation**: computes ionization fraction of a gas in thermal equilibrium.
   Define: Ionization fraction

   $$X \ := \ \frac{n_p}{n_B} \, , \tag{173}$$

   with the number of free protons n_p and the number of baryons n_B.
   The Saha equation reads

   $$\frac{1-X}{X^2} \ \approx \ 3.8 \frac{n_B}{n_\gamma} \left(\frac{k_B T}{m_e c^2} \right) \exp \left[\frac{13.6eV}{k_B T} \right] \, . \tag{174}$$

   R.H.S. small $\Rightarrow X$ close to 1, corresponding to full ionization.
   Define recombination as $X_{rec} = 0.1$, i.e. process is 90 % completed
   $\Rightarrow k_B T_{rec} \approx 0.31 eV \Rightarrow T_{rec} \approx 3600K$.

3. Decoupling happens when the duration of the photon mean free path equals the age of the Universe; the mean free paths grows much faster than the Universe.
   From that one gets $t_{dec} = 3000K$.

7 Lecture 6: How all began

We 28.11.: Chapter 11,12

7.1 The Early Universe

Start from the present and work backwards
Relativistic Particles

- **Photons**

$$\Omega_{rad} = \frac{2.47}{h^2} \cdot 10^{-5} \,. \tag{175}$$

- **Neutrinos** are far too elusive to detect the primordial neutrino background!
  If neutrinos are massless, then we get (Problem 11.1)

$$\Omega_\nu = 3 \cdot \frac{7}{8} \cdot \left(\frac{4}{11}\right)^{\frac{4}{3}} \cdot \Omega_{rad} = 0.68\Omega_{rad} = \frac{1.68}{h^2} \cdot 10^{-5} \,. \tag{176}$$

- Adding them together we get

$$\Omega_{rel} = \frac{4.15}{h^2} \cdot 10^{-5} \,. \tag{177}$$

Non-Relativistic Particles:

$$\Omega_{NR} = \Omega_0 = 0.3 \,. \tag{178}$$

= =
We know the time evolution of the densities:

$$\rho_{rel} = \rho_{0,rel} \frac{a_0^4}{a^4} \,, \tag{179}$$

$$\rho_{matter} = \rho_{0,mat} \frac{a_0^3}{a^3} \,, \tag{180}$$

$$\Rightarrow \frac{\Omega_{rel}}{\Omega_{mat}} = \frac{\rho_{rel}}{\rho_{mat}} = \frac{\rho_{0,rel}}{\rho_{0,mat}} \frac{a_0}{a} = \frac{4.15 \cdot 10^{-5}}{\Omega_0 h^2} \frac{1}{a} \,. \tag{181}$$

Now we can go backwards:

- We have seen that decoupling takes place at $T = 3000K = 1000T(t_0)$, thus we get $a_{dec} = 1/1000a(t_0) = 1/1000$:

$$\Rightarrow \frac{\Omega_{rel}}{\Omega_{mat}} = \frac{0.04}{\Omega_0 h^2} \approx 0.27\,. \tag{182}$$

- Matter and radiation have the same density, if

$$a = a_{eq} = \frac{4.15 \cdot 10^{-5}}{\Omega_0 h^2} = \frac{1}{24096.4\Omega_0 h^2} \approx \frac{1}{3542}\,. \tag{183}$$

  In the **epoch of matter-radiation equality** we expect a temperature of about $3542T_0 = 9652K$.

Now we can determine the full temperature versus time relation for the Universe!

1. Set $k = 0$ (observed) and $\Lambda = 0$ (good approximation in the early Universe).
2. We have always:

$$T \propto \frac{1}{a}\,. \tag{184}$$

3. During matter dominance we have

$$a \propto t^{\frac{2}{3}} \Rightarrow T \propto t^{-\frac{2}{3}}\,. \tag{185}$$

4. Fix the proportionality constant by assuming the age of the Universe to 12 Gyrs (a slight underestimate to correct for the cosmological constant) and the temperature to 2.725K

$$\boxed{\frac{T}{2.725K} = \left(\frac{4 \cdot 10^{17}s}{t}\right)^{\frac{2}{3}}\,.} \tag{186}$$

   This equation holds for times after the matter-radiation equality of equilvalently for temperature $T < T_{eq}$.

5. At the point of matter - radiation equality we have

$$\begin{aligned} T_{eq} &= \frac{2.725K}{a_{eq}} = 65662.7\Omega_0 h^2 K = 9652K\,. && (187) \\ \Rightarrow t_{eq} &= \left(\frac{2.725K}{T_{eq}}\right)^{\frac{3}{2}} 4 \cdot 10^{17} s && (188) \\ &= 1.07 \cdot 10^{11} s \cdot \Omega_0^{-\frac{3}{2}} h^{-3} \left(= 1.9 \cdot 10^{12} s\right) && (189) \\ &\approx 3389 \Omega_0^{-\frac{3}{2}} h^{-3} years \left(= 60131 years\right)\,. && (190) \end{aligned}$$

6. Decoupling happened at $T_{dec} = 3000k$, i.e. in the matter-dominated epoch. Thus we can calculate the time of decoupling with our above formula:

$$\begin{aligned} t_{dec} &= \left(\frac{2.725K}{T_{dec}}\right)^{\frac{3}{2}} 4 \cdot 10^{17} s && (191) \\ &= 1.095 \cdot 10^{13} s && (192) \\ &\approx 346996 years\,. && (193) \end{aligned}$$

7. At temperatures above T_{eq} radiation dominates and we have

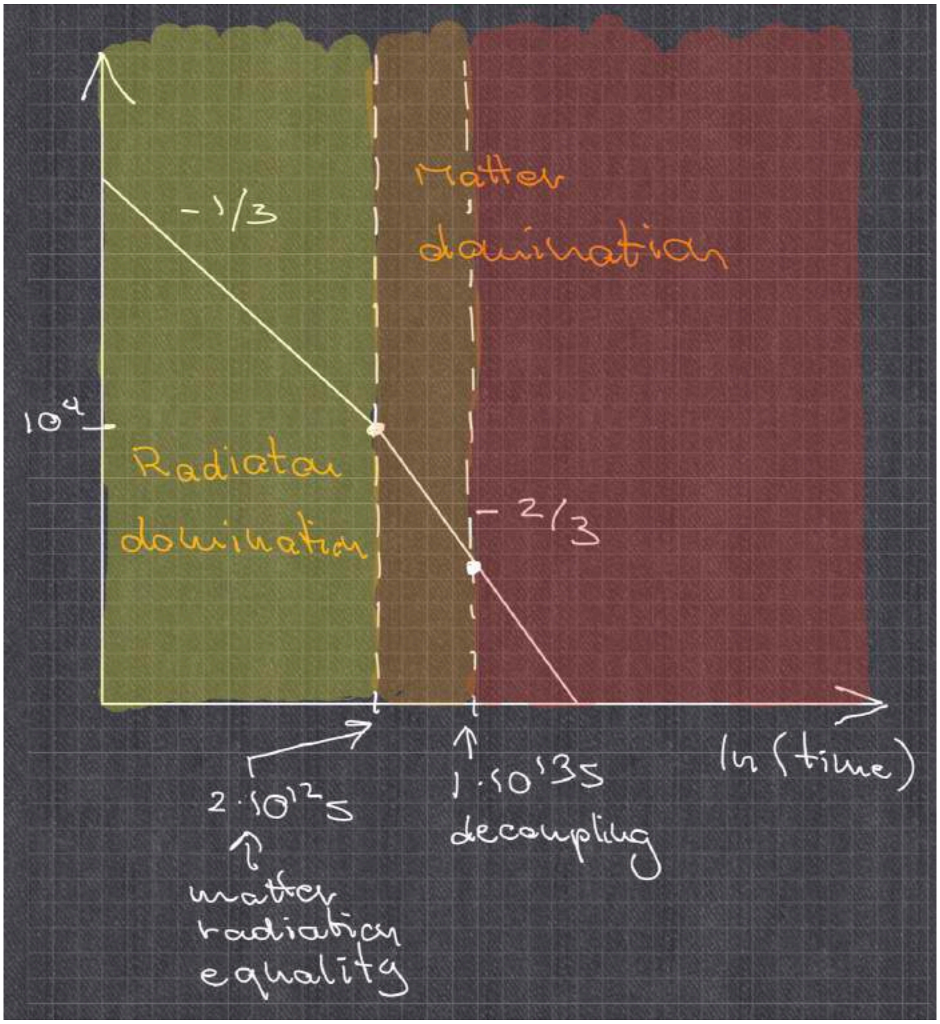
$$\begin{aligned} a &\propto t^{\frac{1}{2}} \Rightarrow \frac{T}{T_{eq}} = \left(\frac{t_{eq}}{t}\right)^{\frac{1}{2}} && (194) \\ \frac{T}{65663\Omega_0 h^2 K} &= \left(\frac{1.07 \cdot 10^{11} s \cdot \Omega_0^{-\frac{3}{2}} h^{-3}}{t}\right)^{\frac{1}{2}} && (195) \\ \frac{T}{9922K} &= 1.35 \cdot 10^6 \left(\frac{1s}{t}\right)^{\frac{1}{2}} && (196) \\ \frac{T}{1.34 \cdot 10^{10} K} &= \left(\frac{1s}{t}\right)^{\frac{1}{2}} && (197) \\ \frac{k_B T}{1.34 \cdot 10^{10} K} &= \left(\frac{1s}{t}\right)^{\frac{1}{2}} && (198) \\ \frac{k_B T}{1.15 MeV} &= \left(\frac{1s}{t}\right)^{\frac{1}{2}} && (199) \end{aligned}$$

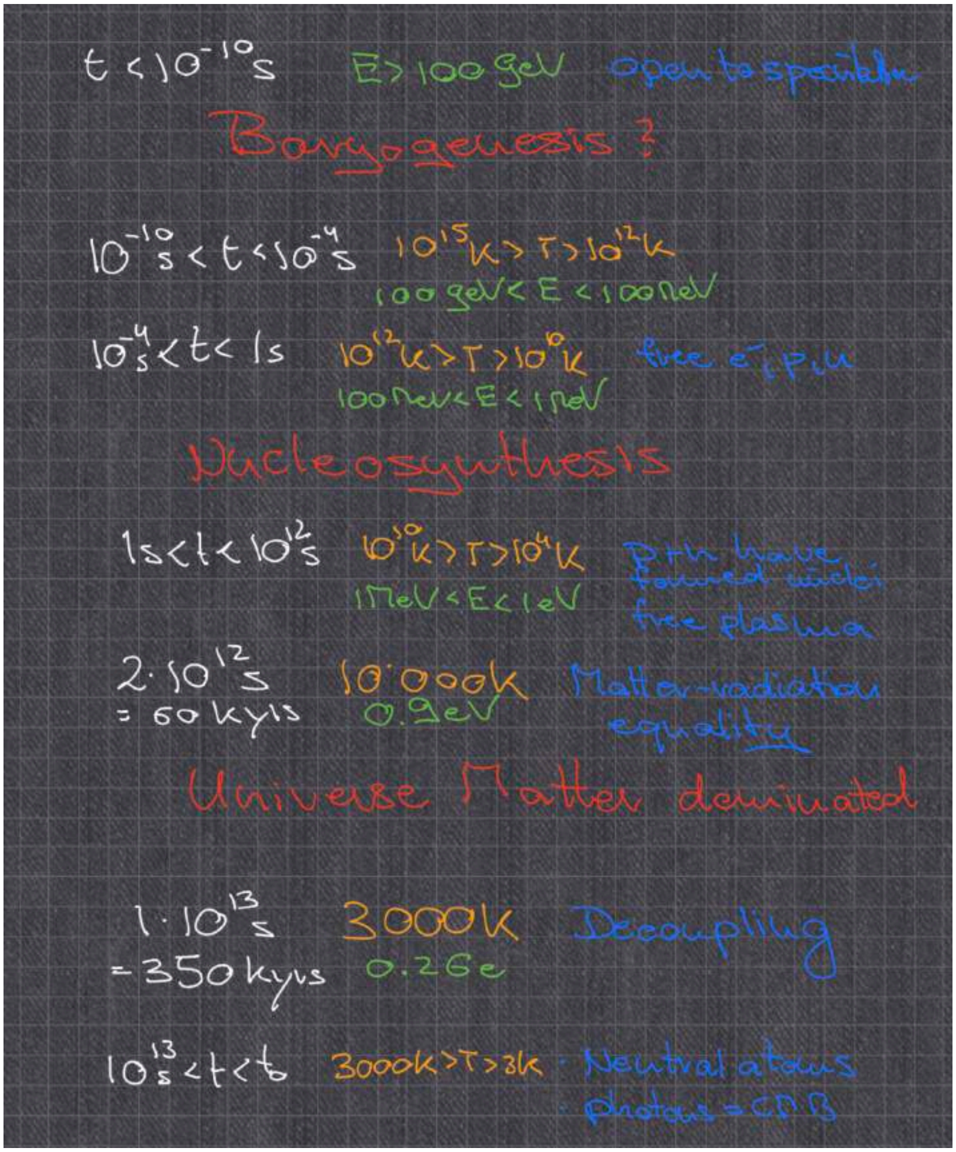
This result can also be directy derived from the Friedmann Equation

$$\begin{aligned}
H^2 &= \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\alpha T^4 & (200)\\
\frac{1}{4t^2} &= \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}\alpha T^4 & (201)\\
\left(\frac{1sec}{t}\right)^{\frac{1}{2}} &= \left(\frac{32\pi G}{3}\alpha\right)^{\frac{1}{4}} T & (202)\\
&= \frac{T}{1.3\cdot 10^{10}K} = \frac{k_B T}{1.1MeV}\,. & (203)
\end{aligned}$$

Thus we have altogether

$$\boxed{\begin{aligned}
\frac{k_B T}{1.1MeV} &= \left(\frac{1s}{t}\right)^{\frac{1}{2}} & \text{for } t < 1.9\cdot 10^{12}s\,. & \quad (204)\\
\frac{T}{2.725K} &= \left(\frac{4\cdot 10^{17}s}{t}\right)^{\frac{2}{3}} & \text{for } t > 1.9\cdot 10^{12}s\,. & \quad (205)
\end{aligned}}$$


-1/3
Matter domination
10^4
Radiation domination
-2/3
2·10^12 s
matter radiation equality
1·10^13 s
decoupling
ln(time)



$t < 10^{-10} s$ $E > 100$ GeV open to speculation

Baryogenesis ?

$10^{-10} s < t < 10^{-4} s$ $10^{15} K > T > 10^{12} K$

$100 \text{ GeV} < E < 100 \text{ MeV}$

$10^{-4}_{s} < t < 1s$ $10^{12} K > T > 10^{10} K$ free e^-, p, n

$100 \text{ MeV} < E < 1 \text{ MeV}$

Nucleosynthesis

$1s < t < 10^{12} s$ $10^{10} K > T > 10^{4} K$ p+n have formed nuclei

$1 \text{ MeV} < E < 1 \text{ eV}$ free plasma

$2 \cdot 10^{12} s$ 10'000K Matter-radiation

= 60 kyrs 0.9eV equality

Universe Matter dominated

$1 \cdot 10^{13} s$ 3000K Decoupling

= 350 kyrs 0.26e

$10^{13}_{s} < t < t_0$ $3000K > T > 3K$ · Neutral atoms

· photons = CMB

Thursday, Nov 29, 14:00 - 15:00
Francesca Calore (LAPTh - CNRS)
Title: ”Indirect dark matter searches: status and perspectives”

7.2 Nucleosynthesis

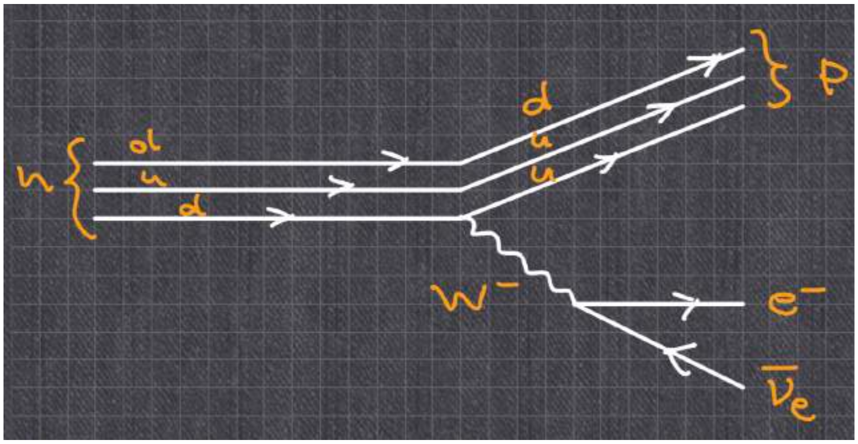
Basic facts:

- proton $m_p c^2 = 938.3$ Mev; neutron $m_n c^2 = 939.6$ MeV

$$\Delta = m_n c^2 - m_p c^2 = 1.3\,\mathrm{MeV}\,. \tag{206}$$

- neutrons decay via the weak decay $n \to p + e^- + \bar{\nu}_e$, their half-life time is $T_{1/2} = 610s$ ($\tau = T_{1/2}/\ln 2 = 880s$).

$$N(t) = N(0)\exp\left[-\frac{t\cdot}{880\,s}\right]\,. \tag{207}$$



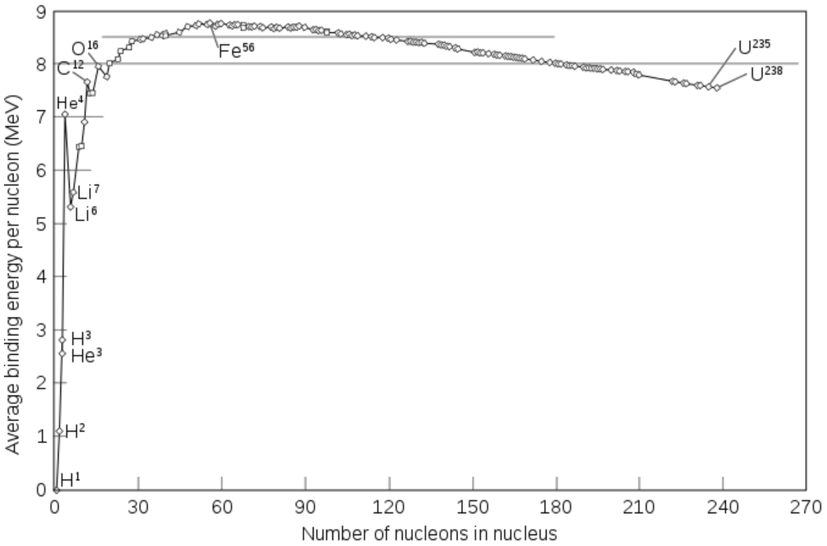
There is currently an interesting decay anomaly observed

$$\tau = \ln 2T_{1/2} = 879.6 \pm 0.6s \;\;(\mathrm{Bottle})\,. \tag{208}$$

$$\tau = \ln 2T_{1/2} = 888.0 \pm 2.0s \;\;(\mathrm{Beam})\,. \tag{209}$$

see e.g. 1811.06546 - the 4 σ discrepancy, might be an indication of dark matter!

- Neutrons bound in a nuclei are stable. The first strongly bound nucleus is the ${}^4_2He$ with a binding energy of about 28 MeV.



The creation of ${}^4_2He$ from protons proceeds via several steps

$$p + n \rightarrow D\,; \tag{210}$$

$$D + p \rightarrow {}^3He\,; \tag{211}$$

$$D + D \rightarrow {}^4He\,; \tag{212}$$

The binding energy of deuterium is 2.2 MeV.

Time evolution of nucleosynthesis

1. At high temperatures p and n are in equlibrium.
2. The Universe cools down until protons and neutrons are non-relativistic but still in thermal equilibrium.
   Their number density is given by the Boltzmann distribution:

$$N \propto m^{\frac{3}{2}} \exp\left(-\frac{mc^2}{k_B T}\right) . \tag{213}$$

Thus we get for the ratio of the number of neutrons and the number of protons

$$\frac{N_n}{N_p} = \left(\frac{m_n}{m_p}\right)^{\frac{3}{2}} \exp\left[-\frac{(m_n - m_p)c^2}{k_B T}\right] \tag{214}$$

As longs as temperatures are high this ratio is close to one - the corresponding reactions are

$$n + \nu_e \rightleftharpoons p + e^- \tag{215}$$

$$n + e^+ \rightleftharpoons p + \bar{\nu}_e \tag{216}$$

3. If temperatures are approaching $\Delta \approx 1.5 \cdot 10^{10}$K, then the ratio will start to deviate significantly from 1. A detailed calculation of the above reaction rates yields that the reactions are getting out of equilibrium at $k_B T = 0.8$ MeV (corresponds to $t = 2s$). At this energy we get

$$\frac{N_n}{N_p} = \left(\frac{m_n}{m_p}\right)^{\frac{3}{2}} \exp\left[-\frac{(m_n - m_p)c^2}{k_B T}\right] \approx \frac{1}{5}\,. \tag{217}$$

4. After this freeze out neutrons will start to decay or be bound in nuclei due to nucleosynthesis, via the decay chains shown above. As soon as the energy falls below $k_B T = 0.06$ MeV, the deuterium is not destroyed anymore and it can proceed into fusioning in helium. $k_B T = 0.06$ MeV corresponds to a time of

$$t = \left(\frac{1.1\text{MeV}}{0.06\text{MeV}}\right)^2 \approx 340s = 5min\,40s\,. \tag{218}$$

This time is amazingly close to the lifetime of the neutron, thus slight changes in this value would have a significant effect on primordial nucleosynthesis. Hence the parameters of the Universe seem to be **fine-tuned**.
At this time the ratio of the number of neutrons to the number of protons is given as

$$\frac{N_n}{N_p} = \frac{1}{5} \exp\left[-\frac{340s}{880s}\right] \approx \frac{1}{7.3}\,. \tag{219}$$

5. Now we simpliy assume that all neutrons tha exist at this time will be bound into ^4He. Then we get for the helium 4 mass density

$$Y_4 = \frac{4N_{He}}{N_p + N_n} = \frac{4 \cdot \frac{N_n}{2}}{N_p + N_n} = \frac{2N_n}{N_n + N_p} = \frac{2}{1 + \frac{N_p}{N_n}} = 0.24 \,. \qquad (220)$$

   This number agrees perfectly with observation!

A more detailed treatment can be found in

V. F. Mukhanov
Nucleosynthesis without a computer
Int. J. Theor. Phys. **43** (2004) 669 [astro-ph/0303073].

Besides ^4He also the abundances of D, ^3He and ^7Li are determined.
Most of the up-to date results are done numerically, recent overview is given in

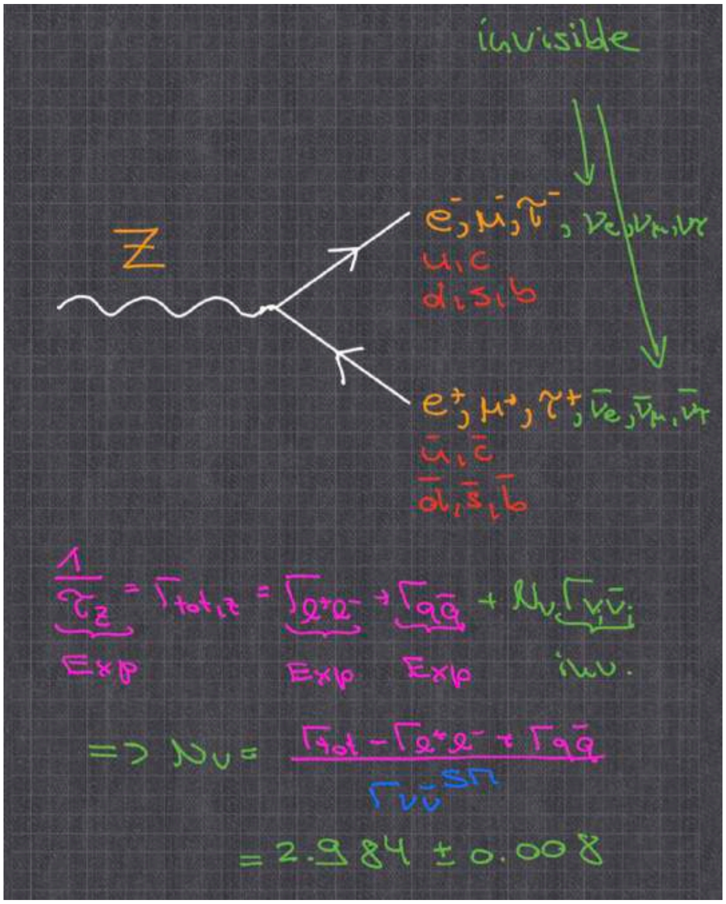
G. Steigman,
Primordial Nucleosynthesis in the Precision Cosmology Era
Ann. Rev. Nucl. Part. Sci. **57** (2007) 463 [arXiv:0712.1100 [astro-ph]].

All in all the comparison of observation with theoretical calculation can be used to determine two important parameter of cosmology:
How to measure the amount of primordial ^{4}He? Tricky, because ^{4}He is also produced in stars ⇒ model dependent determination of Y_4.

1. The number of relativistiv degrees of freedom, in particular the number of light neutrinos. A different value for Ω_{rel} gives a different value for a_{eq}, T_{eq}, t_{eq} and thus a different $T(t)$ expression for the radiation dominated Universe. By comparing observation and theory we can extract the number of neutrino species

$$\begin{aligned} N_\nu &= 2.42^{+0.43}_{-0.41}\,, & (221) \\ N_{nu} &< 3.21 \pm 0.16 \text{ for alternative } ^4\text{He abundance}\,. & (222) \end{aligned}$$

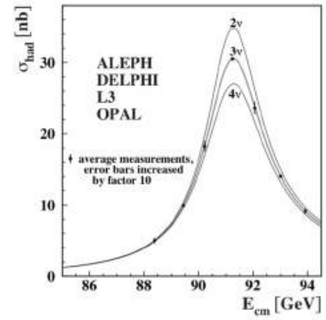
   The number of neutrino species was also determined at LEP.



$$\underbrace{\frac{1}{\tau_Z}}_{\text{Exp}} = \Gamma_{tot,Z} = \underbrace{\Gamma_{e^+e^-}}_{\text{Exp}} + \underbrace{\Gamma_{q\bar q}}_{\text{Exp}} + N_\nu \underbrace{\Gamma_{\nu_i\bar\nu_i}}_{\text{inv.}}$$

$$\Rightarrow N_\nu = \frac{\Gamma_{tot} - \Gamma_{e^+e^-} + \Gamma_{q\bar q}}{\Gamma_{\nu\bar\nu}^{SM}}$$

$$= 2.984 \pm 0.008$$

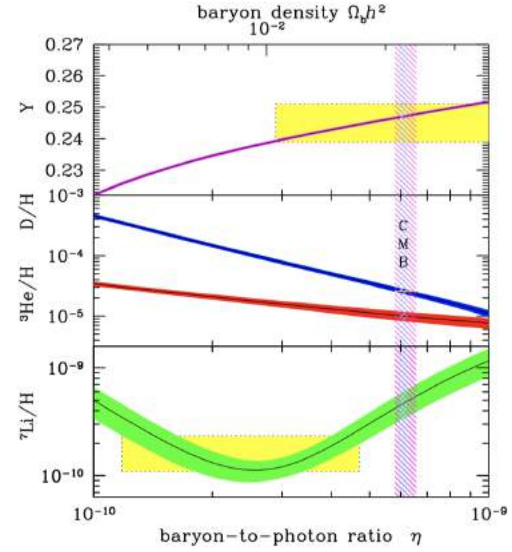


Yielding:

$$N_\nu \quad = \quad 2.984 \pm 0.008\,, \tag{223}$$

2. The density of baryons from which the nuclei are composed. Our simplistic model depended only on Ω_0, a more detailed treatment will depend on Ω_B and yield:

$$0.021 \leq \quad \Omega_B h^2 \quad \leq 0.025\,. \tag{224}$$



Nowadays - with a very high precision - some problems are arising, in particular for ^7Li and ^4He; might be similar to the case of the Hubble parameter.

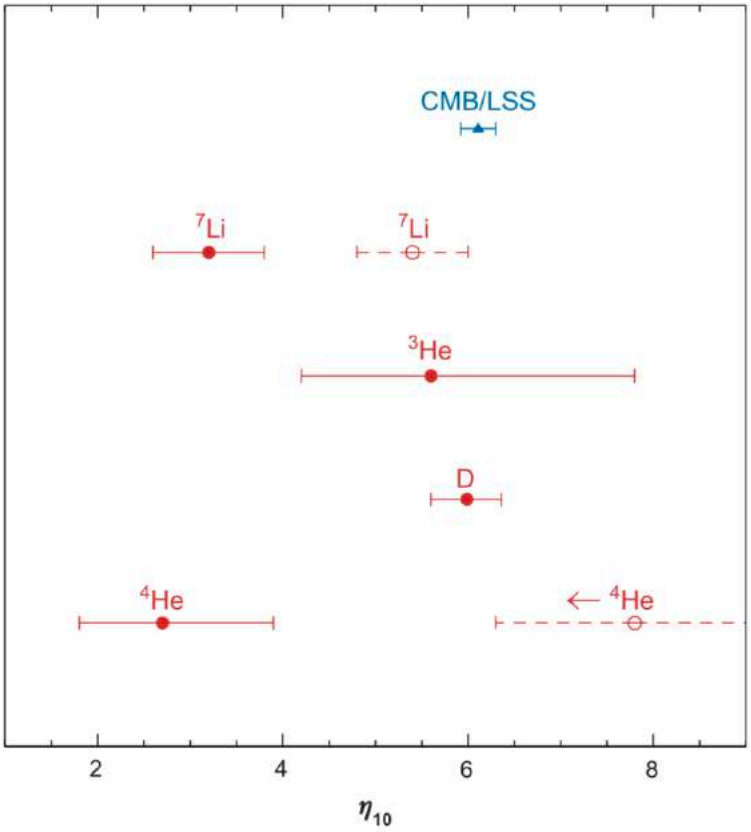


Figure 13

The SBBN-predicted values of η_{10}, and their 1σ uncertainties (*red filled circles*), corresponding to the primordial abundances adopted in Section 3.5, and the non-BBN value inferred from cosmic microwave background radiation (CMB) and large scale structure (LSS) data (*blue triangle*). The open circles and dashed lines correspond to the alternate abundances proposed for ^{4}He and ^{7}Li in Section 3.5.

8 Lecture 7: How it really began

Mo 3.12.: Chapter 13,14

8.1 The Inflationary Universe

Now we move away from well-established and understood topics in cosmology in order to discuss the more speculative idea of inflation that was invented in 1980/81 in order to describe the very, very early times in the Universe. Inflation will somehow create the initial conditions for the Big Bang.

A. H. Guth,
The Inflationary Universe: A Possible Solution to the Horizon and Flatness Problems
Phys. Rev. D **23** (1981) 347
7078 citations counted in INSPIRE as of 02 Dec 2018

A. A. Starobinsky
A New Type of Isotropic Cosmological Models Without Singularity
Phys. Lett. B **91** (1980) 99
3854 citations counted in INSPIRE as of 03 Dec 2018

8.1.1 Problems with the hot Big Bang

1. **The flatness problem:** We have written the Friedmann equation into the following form

$$\Omega_{matter} + \Omega_\Lambda - 1 = \frac{k}{a^2 H^2} \tag{225}$$

$$|\Omega_{tot}(t) - 1| = \frac{|k|}{\dot{a}^2} \,. \tag{226}$$

   - Matter dominance:

$$a \propto t^{\frac{2}{3}} \tag{227}$$

$$\Rightarrow \dot{a} \propto t^{-\frac{1}{3}} \tag{228}$$

$$\Rightarrow |\Omega_{tot}(t) - 1| \propto t^{\frac{2}{3}} \,. \tag{229}$$

   - Radiation dominance:

$$a \propto t^{\frac{1}{2}} \tag{230}$$

$$\begin{aligned} \Rightarrow \dot{a} &\propto t^{-\frac{1}{2}} && (231)\\ \Rightarrow |\Omega_{tot}(t) - 1| &\propto t\,. && (232) \end{aligned}$$

Thus any deviation from $\Omega = 1$ is growing with time. Having today a value close to 1, means we had to have a value very close to one in the beginning, which looks fine-tuned, except there is a dynamical explanation why $\Omega = 1$ or a kind of symmetry.

Fine-tuning for radiation dominated universe: In order to have now ($t_0 = 4 \cdot 10^{17} s$) $|\Omega_{tot}(t_0) - 1| = 0.01$ we would need the following initial conditions:

$$\Rightarrow |\Omega_{tot}(t) - 1| \;=\; |\Omega_{tot}(t_0) - 1| \frac{t}{t_0} \approx 2.5 \cdot 10^{-20} \frac{t}{s}\,. \qquad (233)$$

- $|\Omega_{tot}(t) - 1| = 2.5 \cdot 10^{-7}$ at decoupling ($t \approx 10^{13} s$).
- $|\Omega_{tot}(t) - 1| = 2.5 \cdot 10^{-8}$ at matter radiation equality ($t \approx 10^{12} s$).
- $|\Omega_{tot}(t) - 1| = 2.5 \cdot 10^{-20}$ at nucleosynthesis ($t \approx s$).
- $|\Omega_{tot}(t) - 1| = 2.5 \cdot 10^{-32}$ at electro-weak symmetry breaking ($t \approx 10^{-12} s$).

Remark: if effects of k or Λ become relevant, then our simple estimates above will not hold anymore.

2. **The horizon problem:** Why is the CMB isotropic? The CMB mirco waves coming from the front have travelled since decoupling (almost the age of the Universe) the distance $x \approx 14 GLyrs$. Now the CMB from the back of us has travelled the same distance, but in the opposite direction. Thus the relative distance is $\Delta x \approx 28 GLyrs$, hence the two points cannot have been in causal contact since the Big Bang, but they have more or less the same temperature!
   More sophisticated studies show that the causally connected regions cover only about $\mathcal{O}(1^\circ)$ of the sky. How can then the full sky be in thermal equilibrium?

3. **Relic particle abundance:** GUT theories would create very heavy particles, e.g. magnetic monopoles that freeze out extremely early and will then dominate (*Hitler-complex* according to Kolb/Turner)the expansion of the Universe. There are no indiactions neither directly for such particles nor for the indirect effects of such particles.

8.1.2 Inflationary Expansion

$$\text{INFLATION} \;\Leftrightarrow\; \ddot{a}(t) > 0\,. \tag{234}$$

Looking at the acceleration eqution

$$\frac{\ddot{a}}{a} \;=\; -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{235}$$

we see that inflation requires $\rho c^2 + 3p < 0$. Since densities are positive we need negative pressure

$$p \;<\; -\frac{\rho c^2}{3}\,. \tag{236}$$

Negative pressures can be obtained in particle physics via spontaneous symmetry breaking.
Remember: The cosmological constant could be described as a liquid with $p = -\rho c^2$, yielding an exponential expansion:

$$a(t) \;=\; \exp\left[\sqrt{\frac{\Lambda}{3}}\cdot t\right]\,. \tag{237}$$

After some time inflation must come to an end, with the energy in the cosmological constant being converted into conventional matter.
Typically inflation is supposed to happen at a temperature around $T = 10^{16}$ GeV (GUT scale), corresponding to $t = 10^{-34}s$.

8.1.3 Solving the Big Bang Problems

1. **The flatness problem:**

$$|\Omega_{tot}(t) - 1| \;=\; \frac{|k|}{a^2H^2} \tag{238}$$

$$\;=\; \frac{3|k|}{\Lambda}\exp\left[-\sqrt{\frac{4\Lambda}{3}}t\right]\,. \tag{239}$$

Now Ω_{tot} is forced to values extremely close to one! Inflation predicts thus

$$\Omega_\Lambda + \Omega_0 \;=\; 1\,, \tag{240}$$

which seems to be perfectly fulfilled by observation!

2. **The horizon problem:** Huge expansion of space; the current visible universe was a tiny, causally connected region in the early universe.

3. **Relic particle abundances:** Huge dilution of monopoles et al.

8.1.4 How much Inflation?

How much inflation is needed (i.e. increase in the scale factor during inflation) to explain the current universe?
Assume:

- Inflation ends at $10^{-34}s$.
- Inflation is perfectly exponential.
- After inflation the universe is radiation dominated.
- Ω_{tot} at the beginning of inflation is not hugely different from one.
- Now we have $|\Omega_{tot}(t) - 1| = 0.01$.

According to our previous investigation we will need $|\Omega_{tot}(t) - 1| < 2.5 \cdot 10^{-54}$ at $t = 10^{-34}s$. During inflation H is constant and thus

$$|\Omega_{tot}(t) - 1| \quad \propto \quad \frac{1}{a^2} \,. \tag{241}$$

Thus we need an inflation of the scale factor by a factor of $\sqrt{2.5 \cdot 10^{-54}} = 1.6 \cdot 10^{-27}$, if at the beginning of inflation, the deviation was 1.
If the characteristic expansion time H^{-1} is equal to $10^{-36}s$, when we get in between the times $10^{-36}s$ and $10^{-34}s$ an inflation factor of

$$\frac{a_{final}}{a_{initial}} \quad \approx \quad \exp\left[H(T_{final} - t_{initial}\right] = e^{99} \approx 10^{43} \,, \tag{242}$$

which easily fulfills the required growth!

8.1.5 Inflation and Particle Physics

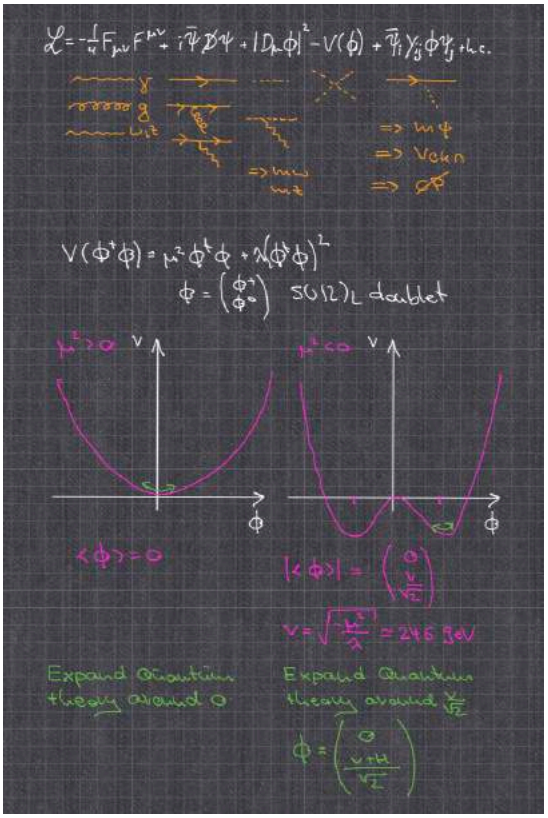
A true model of inflation will give

1. give an explanation of the origin of $\Lambda_{inflation}$

2. give an explanation why inflation started and why it ended.
3. have taken place at higher energies than nucleosynthesis, else this would be spoiled.
   Typically physics is considered that is well above the SM of particle physics!

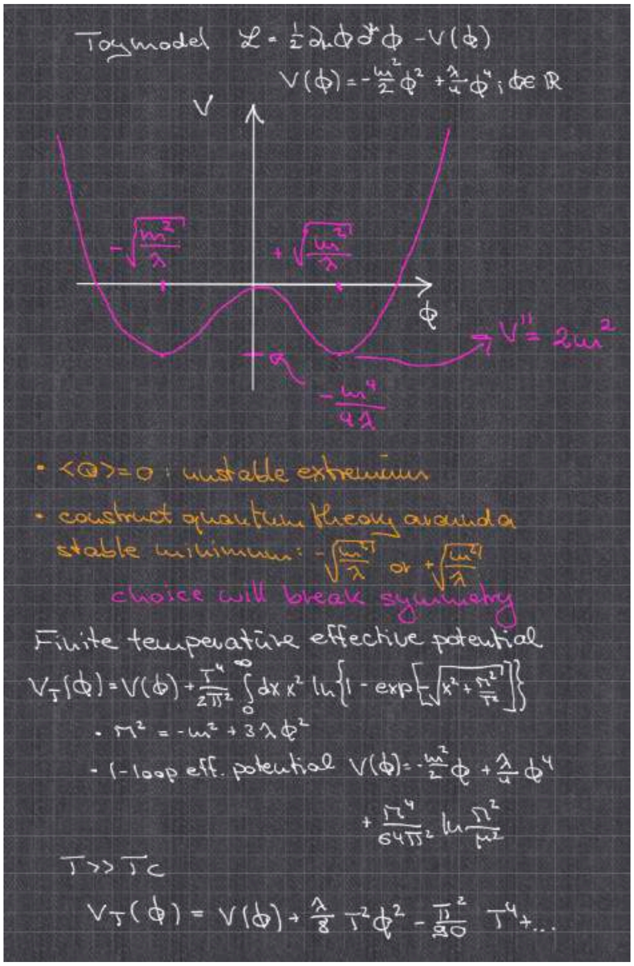
A key feature are phase transitions, which are controlled by **scalar fields**. Scalar fields can have negative pressure and the decay of the scalar field can end inflation.

8.1.6 The scalar sector in the SM



8.1.7 Temperature dependence of scalar fields

The Higgs particle was discovered in 2012 - it is so far the only knwon fundamental scalar particle.



Toymodel $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$

$V(\phi) = -\frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 ; \phi \in \mathbb{R}$

- $\langle \phi \rangle = 0$: unstable extremum
- construct quantum theory around a stable minimum: $-\sqrt{\frac{m^2}{\lambda}}$ or $+\sqrt{\frac{m^2}{\lambda}}$

choice will break symmetry

Finite temperature effective potential

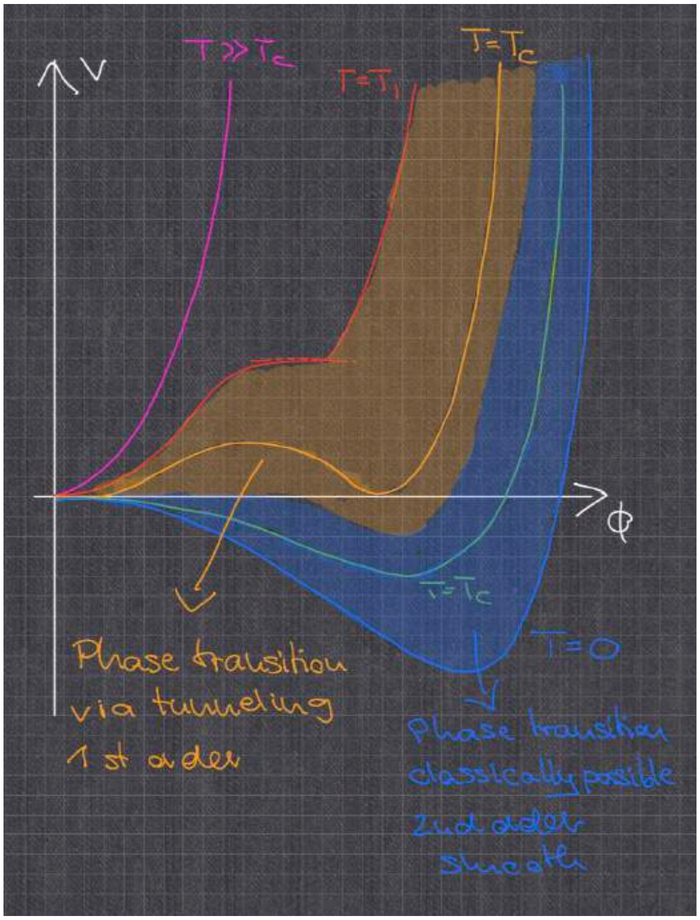
$$V_T(\phi) = V(\phi) + \frac{T^4}{2\pi^2}\int_0^\infty dx\, x^2 \ln\left\{1 - \exp\left[-\sqrt{x^2 + \frac{M^2}{T^2}}\right]\right\}$$

- $M^2 = -m^2 + 3\lambda\phi^2$
- 1-loop eff. potential $V(\phi) = -\frac{m^2}{2}\phi + \frac{\lambda}{4}\phi^4$

$$+ \frac{M^4}{64\pi^2}\ln\frac{M^2}{\mu^2}$$

$T >> T_c$

$$V_T(\phi) = V(\phi) + \frac{\lambda}{8}T^2\phi^2 - \frac{\pi^2}{90}T^4 + \dots$$


$T \gg T_c$
$T=T_1$
$T=T_c$
V
ϕ
$T=T_c$
$T=0$
Phase transition via tunneling 1st order
Phase transition classically possible 2nd order smooth

9 Lecture 8: Inflation in more detail

9.1 GUTs

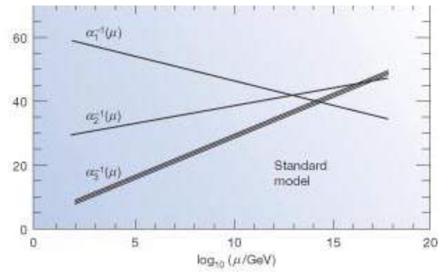
9.1.1 Introduction

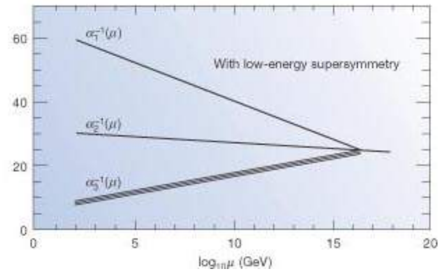
Assume: at high energies (e.g.$T = 10^{16}$ GeV), i.e. in the very early Universe (e.g.$t = 10^{-34}s$), we have a bigger symmetry group G, which contains the SM groups as subgroups (e.g. SU(5) 1974 by Georgi /Glashow). G will be broken at the GUT scale and latest at the electro-weak scale one arrives at the SM gauge groups.

$$\begin{aligned} G &\rightarrow G_1 \times G_2 \times ... \qquad (243) \\ &\rightarrow SU(3)_c \times SU(2)_L \times U(1)_Y \, . \qquad (244) \end{aligned}$$

Motivation for such an approach:

- Gauge coupling unification at about $M \approx 10^{15}$ GeV.





- There is only one gauge interaction and thus only one gauge coupling $\Rightarrow$ reduction of the number of parameter.
- Neutrino mass creation via see-saw gives a similar scale.

9.1.2 Construction of a GUT theory:

1. Choose gauge group G
2. Choose group representation for gauge bosons
3. Choose group representation for fermions
4. Chosse details for symmetry breaking

Historically the first GUT was written down by Pati und Salam in 1973, they considered lepton number as a fourth colour and added in addition a right-handed $SU(2)$. The complete symmetry group was thus a half-simple group (product group):

$$SU(4) \times SU(2)_L \times SU(2)_R \,. \tag{245}$$

The first group contains in the fundamental representation the following particles:

$$\begin{aligned} &(u_r, u_g, u_b, \nu_e)\,, &(246)\\ &(d_r, d_g, d_,e)\,, &(247)\end{aligned}$$

r, g, b indicate colour. $SU(4)$ will be broken to $SU(3)_{QCD}$ and $SU(2)_L \times SU(2)_R$ will be broken down to $SU(2)_L \times U(1)_Y$. This model will give leptoquarks and is currently discussed as a potential solution for the observed flavour anomalies.

The prime candidate for a GUT theory is the $SU(5)$ theory from Georgi and Glashow (1974):

ad 1) SU(5): smallest **simple group** of rank 4, [1] which contains the SM gauge group[2] . Since we have a single gauge group there is also only one gauge coupling g_u.

$$\begin{aligned} D_\mu &= \partial_\mu - i g_u T^a A^a_\mu\,, &(248)\\ \left[T^a, T^b\right] &= i f^{abc} T^c\,, &(249)\\ U &= \exp\left[i\alpha^a T_a\right]\,. &(250)\end{aligned}$$

ad 2) The fundamental representation of $SU(5)$ consists of 5x5 matrices, which can have 50 real parameter and 25 unitarity conditions, plus one condition due to the determinant. Thus 24 free parameter are left. We will choose the matrices in such a way that the upper 3×3 block

---

[1] Rank = Maximal number of simultaneously diagonalisable generators. In the SM there are 2 diagonal generators in SU(3)(λ_3 and λ_8), and one diagonal one in each SU(2) (T_3) and U(1).

[2] Georgi and Glashow have shown that all other groups of rank 4 have problems: $[SU(2)]^4, [O(5)]^2, [SU(3)]^2, [G_2]^2, O(8), O(9), SP8, F_4$.

describes $SU(3)$ and the lower 2×2 block $SU(2)$. Thus we get for the first matrices

$$\lambda^a = \begin{pmatrix} & & & 0 & 0 \\ & \lambda^a & & 0 & 0 \\ & & & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{für} \;\; a = 1, ..., 8 \tag{251}$$

$$\lambda^a = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{1,2} & \\ 0 & 0 & 0 & & \end{pmatrix} \quad \text{für} \;\; a = 9, ..., 10 \tag{252}$$

with the Gell-Mann matrices λ^a and the Pauli matrices σ^i. The relation to the $SU(5)$ parametern is given by

$$T^a := \frac{\lambda^a}{2} \,. \tag{253}$$

Moreover the matrices are normalised

$$tr(\lambda^a \lambda^b) = 2\delta^{ab} \,. \tag{254}$$

The first 8 generators describe gluons, the next two describe the charged weak currents. Besides the two diagonal generators T^3 and T^8 we have two further diagonal ones

$$\lambda^{11} = diag(0, 0, 0, 1, -1) \,, \tag{255}$$

$$\lambda^{12} = \frac{1}{\sqrt{15}} diag(-2, -2, -2, 3, 3) \,. \tag{256}$$

λ^{11} is the 3rd component of the weak isospin (W^3) and λ^{12} describes the weak hypercharge (B).
The 12 remaining generators mix the QCD sector nontrivially with the electro-weak one:

$$\lambda^{13} = \begin{pmatrix} & & & 1 & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad 6 \,\text{possibilities} \;\; a = 13, ..., 18 \tag{257}$$

$$\lambda^{19} \;=\; \begin{pmatrix} & & & i & 0 \\ & 0 & & 0 & 0 \\ & & & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad 6\,\text{possibilities} \;\; a = 19, ..., 24 \tag{258}$$

These 24 matrices fulfill the $SU(5)$-algebra

$$\left[\frac{\lambda^a}{2}, \frac{\lambda^b}{2}\right] = i f^{abc} \frac{\lambda^c}{2}\,, \tag{259}$$

describing 24 gauge bosons: 8 gluons, 3 W-bosons, 1 B-boson and 12 new gauge bosons: 3 X and 3 Y-bosons as well as their anti-particles. Using the notation

$$A_\mu = \sum_a^{24} A_\mu^a \frac{\lambda^a}{\sqrt{2}} \tag{260}$$

one can put all 24 gauge bosons in a single 5 × 5 matrix.

$$A = \begin{pmatrix} G_1^1 - \frac{2B}{\sqrt{30}} & G_1^2 & G_1^3 & \bar{X}_1 & \bar{Y}_1 \\ G_2^1 & G_2^2 - \frac{2B}{\sqrt{30}} & G_2^3 & \bar{X}_2 & \bar{Y}_2 \\ G_3^1 & G_3^2 & G_3^3 - \frac{2B}{\sqrt{30}} & \bar{X}_3 & \bar{Y}_3 \\ X^1 & X^2 & X^3 & \frac{W^0}{\sqrt{2}} + \frac{3B}{\sqrt{30}} & W^+ \\ Y^1 & y^2 & Y^3 & W^- & -\frac{W^0}{\sqrt{2}} + \frac{3B}{\sqrt{30}} \end{pmatrix} \quad \text{für} \;\; a = 1, ..., 8 \tag{261}$$

Where we have defined:

$$W^\pm \;=\; \frac{W^1 \pm i W^2}{\sqrt{2}}\,, \tag{262}$$

$$W^0 \;=\; W^3\,, \tag{263}$$

$$G_\alpha^\beta \;=\; \sum_{i=8}^{8} \frac{G^i \lambda_{\alpha\beta}^i}{\sqrt{2}}\,, \tag{264}$$

$$G_1^1 + G_2^2 + G_3^3 = 0\,. \tag{265}$$

Using the definition of the adjoint representation

$$\left[\frac{\lambda^a}{2}, A_\mu\right] = T_{Adj.}^a A_\mu \tag{266}$$

one can show that the gauge bosons are in the following representations of $SU(3) \times SU(2) \times U(1)$:

$$G_\alpha^\beta \ : \ (8,1,0) \tag{267}$$

$$W^\pm, W^0 \ : \ (1,3,0) \tag{268}$$

$$B \ : \ (1,1,0) \tag{269}$$

$$X, Y \ : \ \left(\bar{3}, 2, -\frac{5}{6}\right) \tag{270}$$

$$\bar{X}, \bar{Y} \ : \ \left(3, \bar{2}, +\frac{5}{6}\right) \tag{271}$$

ad 3) $SU(5)$ has e.g. 5- and 10-dimensional irreducible representations. How can the SM particles be implemented in to that?

Under $SU(3)_c \times SU(2)_L \times U(1)_Y$ we have

$$\begin{array}{rlcc} Q_L : & (3,2,\frac{1}{6})_L & & (3,2,\frac{1}{6})_L \\ u_R : & (3,1,\frac{2}{3})_R & & (\bar{3},1,-\frac{2}{3})_L \\ d_R : & (3,1,-\frac{1}{3})_R & \Rightarrow & (\bar{3},1,+\frac{1}{3})_L \\ L_L : & (1,2,-\frac{1}{2})_L & & (1,2,-\frac{1}{2})_L \\ e_R : & (1,1,-1)_R & & (1,1,1)_L \end{array} \tag{272}$$

In the second column we express everything in terms of left-handed spinors.

The fundamental (5-dimensional) representation of $SU(5)$ is denoted as

$$\psi^\mu = \begin{pmatrix} \psi^\alpha \\ \psi^i \end{pmatrix} \tag{273}$$

with $\mu = 1,2,...,5$, $\alpha = 1,2,3$ and $i = 1,2$. According to our above construction of the $SU(5)$ matrices, $SU(3)$ is only acting on the α-components and $SU(2)$ only on the i-components; hence ψ^α is a 3 dimensional representation of $SU(3)$, i.e. 3 or $\bar{3}$ and ψ^i is a 2 dimensional representation of $SU(2)$, i.e. 2. If one normalises the generator T^{12} properly, one gets for the hyper charge

$$\frac{Y}{2} = diag\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right) \tag{274}$$

and one finds

$$\psi^\alpha = \left(3, 1, -\frac{1}{3}\right) \qquad \psi^i = \left(1, 2, \frac{1}{2}\right) . \tag{275}$$

Thus we get for the fundamental representation of $SU(5)$ under the SM gauge group

$$\begin{aligned} 5 &\equiv \left(3, 1, -\frac{1}{3}\right) \oplus \left(1, 2, +\frac{1}{2}\right) \,, && (276) \\ \bar{5} &\equiv \left(\bar{3}, 1, +\frac{1}{3}\right) \oplus \left(1, 2, -\frac{1}{2}\right) \,. && (277) \end{aligned}$$

(Remark: SU(2) does not have an independent conjugate representation). All in all we have now put 5 SM particles into the fundamental $SU(5)$ representation:

$$\bar{5} = \bar{d}_R, \bar{L}_L \,. \tag{278}$$

Do the missing 10 SM particles fit into the 10 dimensional representation? Having the above decomposition for 5, one can show that the 10-dimensional antisymmetric tensor representation has the following decomposition

$$\begin{aligned} 10 &\equiv \left(\bar{3}, 1, -\frac{2}{3}\right) \oplus \left(3, 2, \frac{1}{6}\right) \oplus (1, 1, 1) && (279) \\ &\equiv \bar{u}_R, Q_L, \bar{e}_R && (280) \\ &\equiv (\psi^{\alpha\beta}, \psi^{\alpha i}, \psi^{ij}) && (281) \\ \psi^{\mu\nu} &= \begin{pmatrix} 0 & \bar{u} & -\bar{u} & d & u \\ -\bar{u} & 0 & \bar{u} & d & u \\ \bar{u} & -\bar{u} & 0 & d & u \\ -d & -d & -d & 0 & \bar{e} \\ -u & -u & -u & -\bar{e} & 0 \end{pmatrix} && (282) \end{aligned}$$

What a wonderful world!

1. All SM particles fit into the $\bar{5}$ and 10 of $SU(5)$ with correct quantum numbers!
2. All anomalies cancel.
3. Charge quanitsation: hypercharge is now fixed. Applying $TrQ = Tr(T_3 + \frac{Y}{2}) = 0$ to $\bar{5}$ one finds $Q_{\bar{d}} = -\frac{1}{3}Q_e$!

What a wonderful world?

1. Leptons and quarks are in the same multiplet, thus the heavy gauge bosons form **Leptoquarks**!
   X-Bosons: transitions among 1-3 and 4, i.e. $\bar{d}$ and ν; charge $\frac{1}{3}$.
   Y-Bosons: transitions among 1-3 and 5, i.e. $\bar{d}$ and e^+; charge $\frac{4}{3}$.
   The coupling term for the fermions in the $\bar{5}$ (ψ^i) and 10 (χ_{ij}) represenatation to the 24 gauge bosons can be written as

$$\begin{aligned} g\bar{\psi}\gamma^\mu A_\mu^T\psi + gTr\left[\bar{\chi}\gamma_\mu\left\{A_\mu,\chi\right\}\right] &= -\frac{g}{\sqrt{2}}W_\mu^\dagger(\bar{\nu}\gamma^\mu e + u_\alpha\gamma^\mu d_\alpha) \\ +\frac{g}{\sqrt{2}}X_{\mu\alpha}^a & \quad \Big[\epsilon^{\alpha\beta\gamma}\bar{u}_\gamma^c q_{\beta\alpha} \\ & \quad \Big[\epsilon^{ab}\left(\bar{q}_{\alpha b}\gamma^\mu e^+ - \bar{l}_b\gamma^\mu d_\alpha^c\right)\Big] \,. \end{aligned} \tag{283}$$

   Besides the SM interactions we have also the following transitions

Transition	ΔB	Force carrier
$X \to u + u$	$\frac{2}{3}$	Diquark
$Y \to u + l^-, \bar{d} + \nu$	$\frac{1}{3}$	Leptoquark

2. Leptoquarks and Diquarks can induce proton decay

$$\tau_p \approx \frac{1}{\alpha_{GUT}}\frac{M_X^4}{m_p^5} = \frac{4\pi}{g_U^2}\frac{M_X^4}{m_p^5}\,. \tag{284}$$

   This is seriously constrained by experiment

$$\tau_p(p \to e^+ + \pi^0) > 10^{33}\ a\,, \tag{285}$$

   which excludes the minimal SU(5) non-SUSY-GUT.

3. Relations among coupling constants: in the SM we have

$$D_\mu \;=\; \partial_\mu + ig_s\sum_{\alpha=1}^{8}G_\mu^\alpha\frac{\lambda^\alpha}{2} + ig\sum_{r=1}^{3}W_\mu^r\frac{\sigma^r}{2} + ig'B_\mu\frac{Y}{2}\,. \tag{286}$$

   In $SU(5)$ we have

$$\begin{aligned} D_\mu &= \partial_\mu + ig_u\sum_{\alpha=1}^{24}A_\mu^\alpha\frac{\lambda^\alpha}{2} & (287) \\ &= \partial_\mu + ig_u\sum_{\alpha=1}^{8}A_\mu^\alpha\frac{\lambda^\alpha}{2} + ig_u\sum_{\alpha=9}^{11}A_\mu^\alpha\frac{\lambda^\alpha}{2} + ig_uA_\mu^{12}\frac{\lambda^{12}}{2} + ig_u\sum_{\alpha=12}^{24}A_\mu^\alpha\frac{\lambda^\alpha}{2}\,. & (288) \end{aligned}$$

Since the Gell-Mann matrices, the Pauli-matrices are normalised in the same way as the $SU(5)$ matrices ($tr(\lambda^a \lambda^b = 2\delta^{ab}$), we get at the unification scale

$$\begin{aligned} g_5 &= g_s = g_3\,. && (289)\\ g_5 &= g = g_2\,. && (290)\end{aligned}$$

A similar relation for the $U(1)$ coupling is not so obvious, we have

$$ig_1\lambda^{12}A^{12}_\mu = ig'YB_\mu\,. \qquad (291)$$

Y is in principle free as its value can be compensated g', $2\lambda^{12}$ is fixed via the normalisation of the $SU(5)$ matrices and one gets

$$g_1 = \sqrt{\frac{5}{3}}g'\,. \qquad (292)$$

In terms of the weak angle we this becomes

$$\begin{aligned} \tan\theta_W &= \frac{g'}{g} = \frac{\sqrt{\frac{3}{5}}g_1}{g_2}\sqrt{\frac{3}{5}}\,, && (293)\\ \Rightarrow \sin^2\theta_W &= \frac{3}{8} = 0.375 \quad \text{vs.}0.22 - 0.23\,. && (294)\end{aligned}$$

4. So far there is no space for a right handed neutrino, it could be implemented via an additional singlet representation.

ad 4) Higgs mechanism: consider two Higgs multiplets acquiring non-vanishing VEVs at different times:

$$SU(5) \xrightarrow{v_1} SU(3)_c \times SU(2)_L \times U(1)_y \xrightarrow{v_2} SU(3)_c \times U(1)_Q\,. \qquad (295)$$

$v_1 \gg v_2$ yiels $M_{X,Y} \gg M_{W,Z}$; proton decay and coupling unification point towards $M_X \approx \mathcal{O}10^{15}$ GeV.
In the **minimal SU(5) model** the scalars H^i_j are in the adjoint (**24**) representation of $SU(5)$ and the scalars ϕ^i are in the vector representation(**5**) - one could in principle also use larger representations.
The most general $SU(5)$-invariant potential[3] of order 4 reads

$$\begin{aligned} V(H,\phi) &= V(H) + V(\phi) + \lambda_4(trH^2)(\phi^\dagger\phi) + \lambda_5(\phi^\dagger H^2\phi)\,, && (296)\\ & V(H) = -m_1^2(trH^2) + \lambda_1(trH^2)^2 + \lambda_2(trH^4)\,, && (297)\\ & V(\phi) = -m_2^2(\phi^\dagger\phi) + \lambda_3(\phi^\dagger\phi)^2\,. && (298)\end{aligned}$$

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[3] We postulated an additional discrete symmetry $H \to -H$ and $\phi \to -\phi$ to get rid of cubic terms.

We have now 7 parameter $\lambda_1, ..., \lambda_5, m_1, m_2$, compared to 2 in the SM. Thus the orginal idea of reducing the number of parameter will not work any more

$$\begin{aligned} (g_1, g_2, g_3, \mu, \lambda) &\rightarrow (g_U, \lambda_1, ..., \lambda_5, m_1, m_2)\,, && (299)\\ 3+1 &\rightarrow 1+7\,. && (300) \end{aligned}$$

1st symmetry breaking: One can show: for $\lambda_2 > 0$ and $\lambda_1 > -7/30\lambda_2)$, $V(H)$ has an extremum at $H = \langle H \rangle$.

$$\langle H \rangle = v_1 \; Diag(2, 2, 2, -3, -3)\,, \quad (301)$$

$$v_1^2 = \frac{m_1^2}{60\lambda_1 + 14\lambda_2}\,. \quad (302)$$

After the first SSB we get the following mass parameter:

Scalar	$SU(3) \times SU(2)$Representation	Mass^2
$(H_8)^\alpha_\beta$	$(8, 1)$	$20\lambda_2 v_1^2$
$(H_3)^r_s$	$(1, 2)$	$80\lambda_2 v_1^2$
H_0	$(1, 1)$	$4m_1^2$
$(H_{X\alpha}, H_{Y\alpha})$	$(3, 2)$	0
$(H^\dagger_{X\alpha}, H^\dagger_{Y\alpha})$	$(\bar{3}, 2)$	0

The 12 massless $H_{X,Y}$ are pseudo-Goldstone bosons and will become the longitudinal degrees of freedom of the 12 X, Y bosons with the mass

$$M_x = M_Y = \sqrt{\frac{25}{2}} g v_1\,. \quad (303)$$

The couplings λ_4 and λ_5 influence $\phi = \phi_t(3, 1), \phi_d(1, 2)$ and we get the masses

$$\begin{aligned} m_t^2 &= -m_2^2 + (30\lambda_4 + 4\lambda_5)v_1^2\,, && (304)\\ m_d^2 &= -m_2^2 + (30\lambda_4 + 9\lambda_5)v_1^2\,. && (305) \end{aligned}$$

2nd symetry breaking: could be provided by the doublett ϕ_d.
Assuming $m_d \ll V_1$, we get

$$V_{eff}(\phi_d) = -m_d^2 \phi_d^\dagger \phi_d + \lambda_3 \left(\phi_d^\dagger \phi_d\right)^2 \quad (306)$$

and

$$\begin{aligned} \langle\phi_d\rangle &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v_2 \end{pmatrix}, && (307) \\ v_2 &= \sqrt{\frac{m_d^2}{\lambda_3}} \approx 250 \ \text{GeV}. && (308) \end{aligned}$$

Now the fermion masses can also be created via Yukawa coupling.

$$f_{AB}^{(1)}(\chi_{A,ij})^T C(\chi_{B,kl})\phi_m \epsilon^{ijklm} + f_{AB}^{(2)}(\chi_{A,ij})^T C\psi_B^i \phi^{j\dagger}\epsilon^{ijklm} + h.c. \quad (309)$$

ψ is the $\bar{5}$ and χ the 10 representation of the fermions.
In the minimal model we get a relation among the masses of leptons and down like quarks

$$\begin{aligned} m_e &= m_d\,, && (310) \\ m_\mu &= m_s\,, && (311) \\ m_\tau &= m_b\,. && (312) \end{aligned}$$

Inlcuding radiative corrections one gets

$$m_b \approx 3m_\tau = 5.3\text{GeV}\,, \quad (313)$$

which is already quite close to reality.

Non one can try to tune the non-fixed parameter of the Higgs potential to fulfill the requirements of inflation.
Cartan investigated all groups with rank larger than 4. Fritzsch and Minkowski found in 1974 that $SO(10)$ contains a 16 dimension spinor representation that could also contain the right-handed neutrino.

$$\begin{aligned} SO(10) &\rightarrow SU(5)\otimes U(1) \rightarrow \text{SM}\,, && (314) \\ 16 &\rightarrow 1\oplus\bar{5}\oplus 10 \rightarrow \text{SM}\,. && (315) \end{aligned}$$

Further possiblities include E_6, E_7, E_8. String theory might indicate $E_8 \times E_8 \rightarrow E_6 \rightarrow SO(10)$

9.2 The scalar sector in cosmology

Now we are coming back to cosmology and check what further requrements we will have on our scalar fields. The **Stress tensor** of a scalar field reads

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{g_{\mu\nu}}{2}\partial_\rho\phi\partial^\rho\phi + g_{\mu\nu}V(\phi)\,. \tag{316}$$

Taking ϕ to be constant, i.e. $\phi = \langle\phi\rangle$, then all derivatives are vanishing and we get

$$T_{\mu\nu} = g_{\mu\nu}V(\langle\phi\rangle)\,, \tag{317}$$

$$\langle T_{00}\rangle = -\frac{m^4}{4\lambda} = \rho_V\,. \tag{318}$$

In the current universe the vacuum energy is

$$\rho_\Lambda = 0.7\rho_c = 0.7\cdot 9.5\cdot 10^{-27}\frac{kg}{m^3} \tag{319}$$

$$= 2.9\cdot 10^{-11}eV^4\,. \tag{320}$$

From the stress tensor we can derive the density ρ and the pressure p

$$\rho = T_{00} = \frac{\dot\phi^2}{2} + \frac{\nabla\phi^2}{2a^2} + V(\phi)\,, \tag{321}$$

$$p = \frac{T_{ii}}{a^2(t)} = \frac{\dot\phi^2}{2} - \frac{\nabla\phi^2}{6a^2} - V(\phi)\,. \tag{322}$$

In a homogoenous background this reduces to

$$\rho = T_{00} = \frac{\dot\phi^2}{2} + V(\phi)\,, \tag{323}$$

$$p = \frac{T_{ii}}{a^2(t)} = \frac{\dot\phi^2}{2} - V(\phi)\,. \tag{324}$$

2 special cases

1. no potential $\Rightarrow \rho = p = \frac{\dot\phi^2}{2} \equiv$ stiff fluid
2. no time dependence $\Rightarrow p = -\rho = V(\phi) \equiv$ cosmological constant

Thus the Friedmann Equation reads

$$H^2 = \frac{8\pi G}{3}\left[\frac{\dot{\phi}^2}{2} + V(\phi)\right] - \frac{k}{a^2}\,. \tag{325}$$

Inserting ρ into the fluid equation we get

$$\dot{\rho} = -3H(\rho + p) \tag{326}$$

$$2\frac{\ddot{\phi}\dot{\phi}}{2} + V'\dot{\phi} = -3H\left(\frac{\dot{\phi}^2}{2} + V(\phi) + \frac{\dot{\phi}^2}{2} - V(\phi)\right) \tag{327}$$

$$\Rightarrow \ddot{\phi} = -3H\dot{\phi} - V'\,, \tag{328}$$

where the first term describes damping and the second one drives evolution. This describes a ball running downhill with friction.
In order to provide a sufficiently long period of inflationary expansion (increase of the scale factor by about e^{100}) we must ensure that the transitition between the two vacua is long enough - this leads to the slow roll scenarios. We can define the slow roll parameter

$$\epsilon = \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2\,, \tag{329}$$

$$\eta = \frac{1}{8\pi G}\frac{V''}{V}\,. \tag{330}$$

Nowadays more often used

$$\epsilon_1 = -\frac{\dot{H}}{H^2} \approx \epsilon \tag{331}$$

$$\epsilon_2 = -\frac{\dot{\epsilon}_1}{H\epsilon_1} \approx 4\epsilon - 2\eta \tag{332}$$

$$\epsilon_3 = -\frac{\dot{\epsilon}_2}{H\epsilon_2} \tag{333}$$

If $\epsilon, \eta \ll 1$ we can neglect $\ddot{\phi}$ and work with the slow roll equation of motion

$$3H\dot{\phi} = -V'\,. \tag{334}$$

Show:

$$\Omega - 1 = \frac{k}{H^2 a^2} \tag{335}$$

$$\frac{d\Omega}{da} = (1 + 3w)\frac{\Omega(\Omega - 1)}{a} \tag{336}$$

Finally: Quantum fluctuation of the scalar field can be the origin of density fluctuations.

9.3 Higgsinflation

But also the SM Higgs is considered as being a candidate for the iunflation: Higgsinflation (review 1807.02376)
F. L. Bezrukov and M. Shaposhnikov,
The Standard Model Higgs boson as the inflaton
Phys. Lett. B **659** (2008) 703 [arXiv:0710.3755 [hep-th]].
1167 citations counted in INSPIRE as of 04 Dec 2018

Problem: making inflation sufficiently long without creating too large density perturbations

9.4 The Initial Singularity

Hawking and Penrose have proven in 1970 that under strong energy condition

$$\rho c^2 + 3p \geq 0 \tag{337}$$

there was an initial singularity.
Nowadays situation less clear:

1. We have a cosmological constant, i.e. $p = -\rho$!
2. For very early times a theory of Quantum Gravity has to be considered!

$$E_{Pl} = \sqrt{\frac{\hbar c^5}{G}} = 1.22 \cdot 10^{19} GeV\,, \tag{338}$$

$$G = \frac{1}{m_{Pl}^2} \text{ in natural units}\,, \tag{339}$$

$$t_{Pl} = \sqrt{\frac{\hbar G}{c^5}} = 5.39 \cdot 10^{-44} s\,. \tag{340}$$

10 Acknowledgements

I would like to thank

References

[1] T. Jubb, M. Kirk and A. Lenz,
Charming Dark Matter,
JHEP **1712** (2017) 010 [arXiv:1709.01930 [hep-ph]].
3 citations counted in INSPIRE as of 28 Nov 2018